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PLANE AND SPHERICAL

TRIGONOMETRY.

BY

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PLANE TRIGONOMETRY.

PREFACE.

THIS book originated in a desire to give to my private pupils a better idea of the trigonometric functions of obtuse angles than they could obtain from any book with which I was acquainted.

I believe that a student can get a comprehensive and thorough knowledge of trigonometry most quickly and easily, if, at the outset, such definitions are given to the trigonometric functions as will apply to all angles: with this idea for a basis, I have endeavored to prepare an elementary text-book for general use. By beginning with an explanation of the use of the negative sign as applied to lines and angles, and then giving general definitions to the trigonometric functions, I have been able to demonstrate all the fundamental formulas in a perfectly general yet simple manner.

While I have tried to present the subject from an elementary point of view, I have not lost sight of the fact, that, to most students, trigonometry is merely a stepping-stone to something higher; and for this reason I have also tried to present the results in such a light as will make them effective tools for the student in his future work.

During the preparation and printing of the book, Prof. J. M. Peirce has kindly given me the benefit of numerous suggestions and criticisms, which have added greatly to its value. Appendix I. is entirely due to him.

Most of the examples contained in the book have been selected from other trigonometries, especially those of Todhunter, Snowball, and Hamblin Smith.

H. N. W.

CAMBRIDGE, MASS., Dec. 3, 1876.

PREFACE TO THE REVISED EDITION.

IN this edition all the typographical errors thus far detected have been corrected; a few changes and additions have been made in the examples of right triangles; and answers to most of the examples have been inserted at the end of Appendix II.

A treatise on Spherical Trigonometry has been prepared by the same author, and now constitutes the second part of this volume.

H. N. W.

CAMBRIDGE, MASS., Sept. 16, 1879.

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PLANE TRIGONOMETRY.

CHAPTER I.

DEFINITIONS AND FIRST PRINCIPLES.

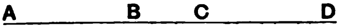
§ 1. TRIGONOMETRY, as the name itself indicates, was originally simply a method of *measuring triangles* by numerical computation from any sufficient data. This method involves the use of certain functions of the angles, which we call the *trigonometric functions*. Modern trigonometry embraces primarily the complete theory of these functions in their most general relations; while the measurement of triangles, or trigonometry proper, is reduced to a mere branch, or application, of the theory.

That part of trigonometry which is occupied with the investigation of the general properties of the trigonometric functions is called by some writers *Angular Analysis*, or *Goniometry;* but it is to be observed that these properties, as well as the very conception of the trigonometric functions themselves, are derived from the doctrine of similar triangles, and the whole subject (at least in its elements) may be regarded as a development of that doctrine.

We must begin our exposition of trigonometry by treating of certain principles adopted in the measurement of lines and angles, the two kinds of geometric quantity which will enter into our formulas.

§ 2. *Measurement of lines.* — A line may be regarded as traced by the movement of a point. Thus we may regard the straight line AD (Fig. 1) as having been traced by the movement of a point from A, the beginning of the line, to D, the end of the line.

FIG. 1.



Straight lines may be added by placing the beginning of the second upon the end of the first (so as to form a continuous straight line), the beginning of the third upon the end of the second, and so on. The straight line contained between the beginning of the first and the end of the last is a line equal to the sum of the lines to be added. Thus (Fig. 1) $AD = AB + BC + CD$.

§ 3. *a.* There are many problems involving straight lines, in which it is neither necessary nor profitable to take into consideration any thing except the lengths of these lines. There are, on the other hand, many cases where it will be useful to take into account direction as well as length.

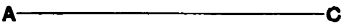
b. If we define any arbitrary direction to be positive, then the opposite direction will be negative. For, if AB is any line, $-AB$ can only be that line which added to AB, by the method of § 2, produces 0; and that is the same line as AB taken in the opposite direction.

The length of a straight line has no algebraic sign: it is always the same, whatever be the position or direction of the line.

c. The sign $+$ or $-$ prefixed to a number or letter denoting a line has reference merely to direction. Thus, if a is positive, the line $+a$ is a line whose direction is positive, and whose length is a units; and the line $-a$ is a line whose direction is negative, and whose length is, as before, a units. But, if a is negative, $+a$ is a line whose direction is negative, and $-a$ is a line of the same length, whose direction is positive. The terms "positive" and "negative" are used merely to enable us to distinguish between two opposite directions; and we can in any figure assume *any directions* (not opposite) to be positive, but, having once assumed certain directions to be positive, the opposite directions must be negative. (*v.*, however, § 20, *d.*)

d. It is possible to represent a straight line completely by an expression involving the points at its extremities. Thus AC (Fig. 2) is an expression for the straight line which joins the points A and C, whose length is the distance between the points, and whose direction is the direction from A, the point indicated by the first letter, to C, the point indicated by the second letter. $-$AC denotes a line having the same length as the above, but an opposite direction, but this latter line is the line CA. $\therefore$ CA $= -$AC, or AC $+$ CA $= 0$. The truth of this equation is evident if, as in § 2, we regard a line as traced by the movement of a point. For if a point moves from A to C, and then from C back to A, it is evident that its final distance from the starting-point will be nothing.

FIG. 2.



Similarly in each of the figures 3, 4, and 5, we have, —

$$\text{AB} + \text{BC} + \text{CA} = 0$$

$$\text{BC} + \text{CA} + \text{AB} = 0$$

$$\text{CB} + \text{BA} + \text{AC} = 0$$

FIG. 3.

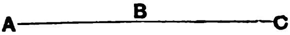


FIG. 4.

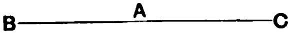
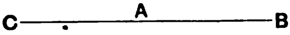


FIG. 5.



This principle may be extended to any number of points in the same straight line.

§ **4.** *Measurement of angles.*—A right angle is divided into 90 equal parts called degrees; a degree is subdivided into 60 minutes; and a minute, into 60 seconds. The degree, minute, and second are distinguished by the characters ° ′ ″ respectively. A degree of arc is $\frac{1}{360}$ of the circumference of the circle to which the arc belongs. It evidently has different lengths for different circles. The degree of arc is subdivided in the same manner as the degree of angle. $\frac{1}{4}$ the circumference, or the arc intercepted by the sides of a right angle whose vertex is at the centre, is an arc of 90°, and is called a *quadrant.* The same term is applied to the sector bounded by two radii at right angles to each other. The four divisions of a plane separated by two lines perpendicular to each other are also occasionally called quadrants.

Central angles contain the same number of degrees as their intercepted arcs.

§ **5.** The *complement* of an angle or arc is the remainder obtained by subtracting the angle or arc from 90°.

The *supplement* of an angle or arc is the remainder obtained by subtracting the angle or arc from 180°.

Thus the complement of 25° is 65°; the supplement of 25° is 155°.

Two angles or arcs are therefore complements of each other when their sum is 90°; and they are supplements of each other when their sum is 180°.

According to these definitions, the complement of an angle or arc that exceeds 90° is negative. Thus the complement of 100° is $90° - 100° = -10°$. In like manner the supplement of 200° is $180° - 200° = -20°$.

§ **6.** *Circular measure of an angle.*

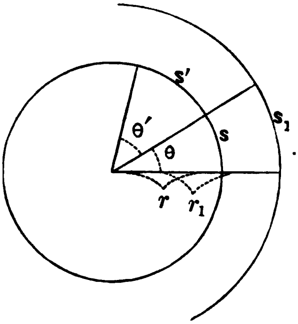
As has been indicated above, an angle may be measured in degrees and subdivisions of a degree. There is, however, another method of measurement in use, called the *Circular Method*, by which an angle is measured by the ratio of its arc in the circle described from its vertex as a centre to the radius of that circle. To show that this method is legitimate, it must be proved that *any two angles are proportional to the ratios by which we propose to measure them.*

Now, in the same circle, or equal circles, angles at the centre are proportional to their intercepted arcs. Therefore (Fig. 6), —

$$\theta : \theta' = s : s' \qquad \text{(I.)}$$

$$\therefore \theta : \theta' = \frac{s}{r} : \frac{s'}{r} \qquad \text{(II.)}$$

FIG. 6.



But, since similar arcs are proportional to their radii, we have $\frac{s}{r} = \frac{s_1}{r_1}$; that is, the ratio of *intercepted arc to radius* is the same for the same angle, whatever the value of the radius. Substituting this value of $\frac{s}{r}$ in equation II. above, we get $\theta : \theta' = \frac{s_1}{r_1} : \frac{s'}{r}$. *Q.E.D.*

§ **7.** Corollary. Any quantity can be put equal to its measure. We can therefore put $\theta = \frac{s}{r}$. Now when the radius (r) is unity, we have $\theta = \frac{s}{1} = s$; therefore the circular measure of an angle is equal to *the length of the intercepted arc in a circle whose radius is unity.*

§ 8. *Angular unit of circular measure.*

As in § 7 we can put $\theta = \frac{s}{r}$. Making $s = r$ we shall have $\theta = \frac{r}{r} = 1$. Therefore the angular unit of circular measure is *that angle whose intercepted arc is equal to the radius.*

§ 9. *To change from circular measure to degree measure, and the converse.*

We will first find how many degrees there are in the angular unit of circular measure; i.e., we will reduce θ to degrees when $s = r$. The ratio of *circumference* to *diameter* is constant for all circles, and is approximately equal to 3.1415927, which is represented by the Greek letter π (*v.* Plane Geometry). Therefore letting C, D, and R, represent circumference, diameter, and radius respectively, we have $\frac{C}{D} = \pi$; $\therefore$ $C = \pi D$; but $D = 2R$; $\therefore$ $C =$ arc $360° = 2\pi R$. In the same circle, or equal circles, central angles are proportional to their intercepted arcs: —

$$\therefore \frac{\text{angle } \theta}{\text{angle } 360°} = \frac{\text{arc S}}{\text{arc C}} = \frac{S}{2\pi R} = (\text{when } S = R)\ \frac{R}{2\pi R} = \frac{1}{2\pi}$$

$$\therefore \text{ angular unit} = \frac{360°}{2\pi} = \frac{180°}{\pi} = \frac{180°}{3.1415927}$$
$$= 57°.2957795.$$

Conversely, the units of degree-measure have the following values in circular measure; and these values are also the lengths of the corresponding arcs of the unit circle: —

$$\text{circ. meas. } 1° = \frac{\pi}{180} = 0.01745329,$$

$$\text{circ. meas. } 1' = \frac{1°}{60} = 0.00029089,$$

$$\text{circ. meas. } 1'' = \frac{1'}{60} = 0.00000485.$$

To convert an angle from one system to the other, we have only to multiply the given measure of the angle by the value of its unit in the new system. For example, if x is the circular measure of an angle, its value in degrees is

$$x \times 57.296\ldots.;$$

and the circular measure of x° is

$$x \times 0.01745\ldots;$$

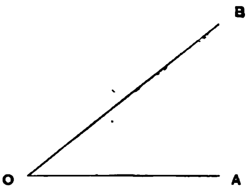
§ **10.** $\text{Angle } 360^\circ = \frac{\text{arc } 360^\circ}{R} = \frac{2\pi R}{R} = 2\pi,$

$$\text{Angle } 180^\circ = \ldots\ldots\ldots\ldots\ldots\ldots \pi,$$
$$\text{Angle } 90^\circ = \ldots\ldots\ldots\ldots\ldots\ldots \tfrac{1}{2}\pi,$$
$$\text{Angle } 45^\circ = \ldots\ldots\ldots\ldots\ldots\ldots \tfrac{1}{4}\pi.$$

The expressions 2π, π, $\frac{1}{2}\pi$, and $\frac{1}{4}\pi$ are often used to denote the angles which they measure.

§ **11.** *General Angular Magnitude.* "A clear notion of the magnitude of an angle will be obtained by supposing that one of its sides as OB (Fig. 7) was at first coincident with the other side OA, and that it has revolved about the point O (turning upon O as the leg of a pair of dividers turns upon its hinge) until it has arrived at the position OB."—*Chauvenet's Geometry.*

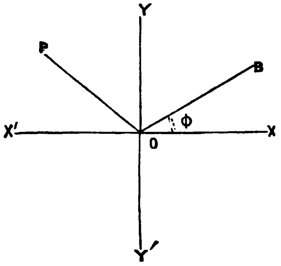
FIG. 7.



We can, then, regard an angle as the measure of the amount of the rotation of a line which turns in a plane about one of its own points. The angle AOB (Fig. 7), for example, shows how much the line OB has rotated from its original direction OA about O as a centre.

§ **12.** Since there is no limit to the rotation of a line about one of its points, there is no limit to angular magnitude.

FIG. 8.

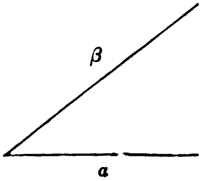


Let XX′ and YY′ (Fig. 8) be perpendicular to each other at the point O. Let OX be the original position of OB. When OB has moved to the position OY, it makes with OX an angle of 90°; when to the position OX′, an angle of 180°; when to the position OY′, an angle of 270°; when to OX again, an angle of 360°; if it continues to move to OY, an angle of $360° + 90°$, or 450°; and, as long as OB continues to move, the angle XOB continues to increase. OX, the original position of the line OB, is called the *initial* line with respect to the angle XOB, while OB is called the *terminal* line with respect to this angle.

§ **13.** When we represent an angle by three letters, we shall always denote the vertex by the middle letter, a point through which the initial line passes by the first letter, and a point through which the terminal line passes by the third letter.

Prof. Peirce has made use of the following angular notation: Let α and β (Fig. 9) denote two lines. The angle which β makes with α, or the angle *alpha-beta*, as it is called, is denoted by the expression $\overset{\beta}{\alpha}$, the lower letter denoting the initial line, and the upper letter the ter-

FIG. 9.



minal line. Such symbols must be carefully distinguished from fractions. This notation may be used even in the case of lines which do not intersect.

§ **14.** Rotation as considered in § 12 (i.e., opposite that of the hands of a clock) is called positive rotation. Rotation in the contrary direction is therefore negative rotation. An angle is positive or negative according as it is conceived to be generated by positive or negative rotation. Thus, we may take $BOX = -XOB$, $\frac{\alpha}{\beta} = -\frac{\beta}{\alpha}$.

§ **15.** Let AA′ and B′B″ (Figs. 10–13) be two straight

FIG. 10.

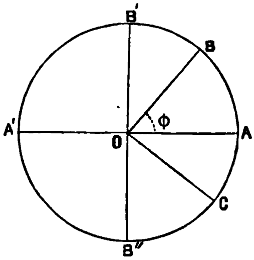


FIG. 11.

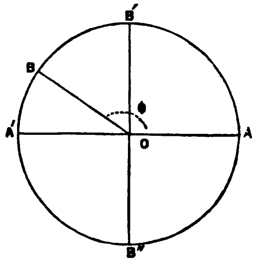


FIG. 12.

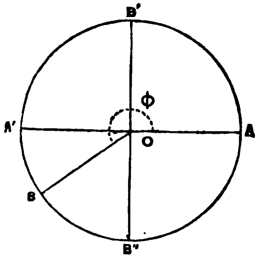
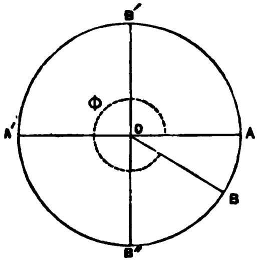


FIG. 13.



lines perpendicular to each other at O, the centre of the circle of which OA is the radius. The arcs AB′, B′A′, A′B″, and B″A, or the sectors AOB′, B′OA′, A′OB″, and B″OA, are quadrants (*v.* § 4). In accordance with what has been said about positive rotation (*v.* § 14), we will call the sector AOB′ (or the arc AB′) the first quadrant, B′OA′ the second quadrant, A′OB″ the third quadrant, and B″OA the fourth quadrant. Therefore the angle marked φ in the figures 10, 11, 12, and 13, is an angle of (i.e. terminating in) the 1st, 2d, 3d, and 4th quadrants, respectively.

§ **16.** Angles in the same plane may be added by first placing their vertices at the same point, and then placing the initial line of the second upon the terminal line of the first, the initial line of the third upon the terminal line of the second, and so on. The angle made by the terminal line of the last with the initial line of the first is an angle equal to the sum of the angles to be added. Thus (Fig. 8) XOP = XOB + BOY + YOP. On the same principle, if α, β, γ, δ, &c., are any lines in one plane, whether intersecting at one point or not, we have:—

$$\genfrac{}{}{0pt}{}{\beta}{\alpha}+\genfrac{}{}{0pt}{}{\gamma}{\beta}+\genfrac{}{}{0pt}{}{\delta}{\gamma}=\genfrac{}{}{0pt}{}{\delta}{\alpha}.$$

§ **17.** The angle AOB (Figs. 12, 13) may have been formed either by the positive or negative rotation of OB; we can therefore either consider it to be a positive angle having a number of degrees equal to 180° + A′OB, or a negative angle having a number of degrees equal to 180° — A′OB. These are, in fact, only two values, which differ by 360° (*v.* § 18). The angle AOB″ also (Figs. 10–13) can be considered equal to + 270° (180° + A′OB″), or equal to — 90° [—(180°— A′OB″)].

In Fig. 10, where OC is supposed to be perpendicular to OB, we have the angle AOC $= \varphi - 90°$. This equation indicates that if the rotating line rotates in the positive direction by an amount equal to φ (i.e., to OB), and then in the negative direction by an amount equal to 90°, it will finally take the position OC, and will make with the initial line (OA) the angle AOC, which will be positive or negative, according as φ is greater or less than 90°.

§ **18.** If *to* the angle φ (Figs. 10–13), we add 360° once, twice, or any integral number of times, the resulting angle will still be bounded by the lines OA and OB. If *from* the angle φ we subtract 360° once, twice, or any integral number of times, the resulting angle will still be bounded by the lines OA and OB.

The expression AOB, then, may represent any one of a series of angles; one angle of this series is the angle φ, and all the other angles of the series differ from it by some multiple of 360°. Thus $\text{AOB} = \varphi + k\,360°$, where k stands for 0, or any integer positive or negative. This principle is often important in the addition of angles.

§ **19.** Bearing in mind what has been said about the addition of angles (§ 16), and also the contents of the preceding section, we see at once the truth and significance of the following equations obtained from figures 14 and 15.

FIG. 14.

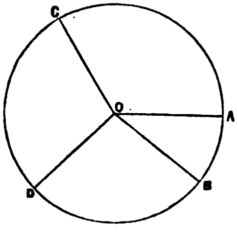
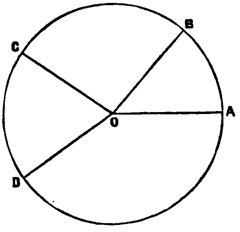


FIG. 15



$$\begin{gathered}\text{AOD} + \text{DOA} = k\ 360^\circ,\\ \text{AOD} + \text{DOC} + \text{COB} + \text{BOA} = k\ 360^\circ,\\ \text{AOB} + \text{BOD} + \text{DOC} + \text{COA} = k\ 360^\circ,\\ \text{AOC} + \text{COB} + \text{BOD} + \text{DOA} = k\ 360^\circ.\end{gathered}$$

where k, as in § 18, denotes any integer, positive, negative, or null.

In like manner, if α, β, γ, δ, are any lines in one plane,—

$$\genfrac{}{}{0pt}{}{\beta}{\alpha} + \genfrac{}{}{0pt}{}{\gamma}{\beta} + \genfrac{}{}{0pt}{}{\delta}{\gamma} + \genfrac{}{}{0pt}{}{\alpha}{\delta} = k\ 360^\circ$$

§ **20.** Before proceeding further we must lay down rules concerning the directions of certain lines which we shall have to treat of in their relations to angles.

a. When speaking of an angle, we shall always regard the vertex as a point of reference or origin with respect to this angle; and we will define the directions of the initial and terminal lines (§ 12) to be always positive, when considered relatively to the angle. Thus, when we speak of the angle between two lines, we always mean the angle between their positive directions.

FIG. 16.

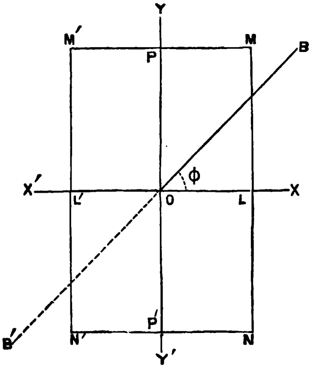
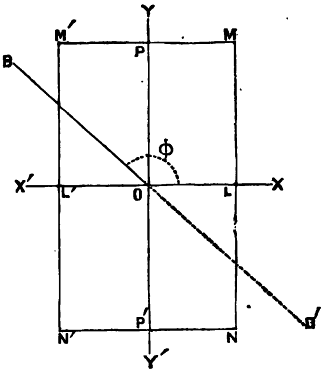


FIG. 17.



In each of the figures 16, 17, 18, and 19, for example, O, the vertex of the angle φ, will be an origin, and the directions of the initial and terminal lines OX and OB will be positive.

FIG. 18.

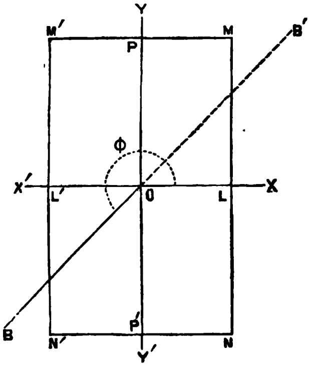
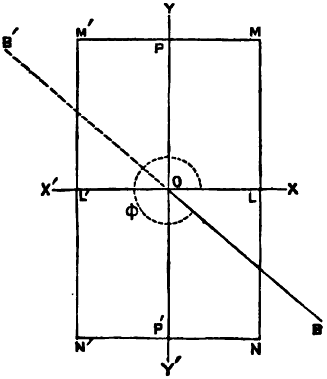


FIG. 19.

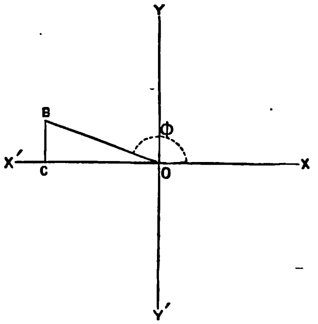


b. Through O, making with the initial line OX an angle of + 90°, draw OY. The lines XX′ and YY′ are called *axes*. We define OX and OY as the positive directions of the axes; OX′ and OY′ are therefore their negative directions. These are also the positive and negative directions of all lines parallel to the axes, when considered relatively to them. Thus PM and P′N are positive, while PM′ and P′N′ are negative; and LM and L′M′ are positive, while LN and L′N′ are negative. Relatively to the angle XOB, OB is positive, and OB′ negative; but relatively to the angle XOB′, OB is negative, and OB′ positive.

c. Whenever in the same problem we have to treat of two or more angles whose vertices and initial lines do not coincide, we can either refer all the angles to the same origin and to the same initial line, or we can regard each vertex as an origin, and each initial and terminal line as positive, *with respect to the angle to which it belongs.* If the latter method is adopted, care must be taken not to confound a direction which is positive with respect to one angle with a direction which is positive with respect to a different angle; for these directions *may* be, when compared with one another, directly opposite.

d. When we have $\varphi = 180°$, or $\varphi = 270°$, the terminal line (OB) will have the negative direction of the axis: we shall, however, still consider OB to be positive, when regarded as the terminal line of φ. Suppose, for example, we have the right triangle BOC (Fig. 20). The base OC is negative; and the hypothenuse OB and perpendicular CB are positive. Now, suppose φ to increase until OB coincides with OC. We shall now regard our right triangle to be one whose perpendicular is zero, whose base OC is negative, and whose hypothenuse OB is, as before, positive.

FIG. 20.



§ **21.** The following principles of Plane Geometry must be borne in mind: —

a. The sum of the angles of a plane triangle is equal to 180°.

b. The sum of the two acute angles of a plane right triangle is equal to 90°. Either of these angles is then the complement (§ 5) of the other.

c. When two plane right triangles have an acute angle and a side of the one equal to an acute angle and the homologous side of the other, the triangles are equal, and all of their homologous parts are equal.

d. When two plane right triangles have an acute angle of the one equal to an acute angle of the other, the triangles are similar, and their homologous sides are therefore proportional.

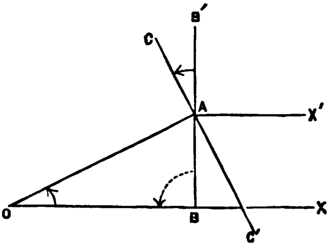
e. The square on the hypothenuse of a right triangle is equal to the sum of the squares on the other two sides.

f. In any triangle the greater side is opposite the greater angle, and the converse.

g. The sum of two sides of a triangle is greater than the third side.

h. Two angles are equal when their sides are respectively perpendicular; but we must be careful to take the sides of the respective angles in the same order, and to measure the angles in the same direction. (*v.* § 14.) In Fig. 21, for example, the angle B′AC = angle XOA, for both angles are measured in the positive (§ 14) direction, and B′AC is formed by AB′, which is perpendicular to OX, and AC, which is perpendicular to OA. Similarly, angle BAC′ = XOA, and angle X′AB′ = ABO.

FIG. 21.



i. The chord of an arc of 60° is equal to the radius.

EXAMPLES.

1. To what quadrant does each of the following angles belong? 289°, 368°, 510°, 640°, 1178°, — 80°, — 188°, — 1722°.

2. Represent by figures the following angles, where in each case $\varphi = 60°$, and $\theta = 30°$; $\varphi - 360°$, $\varphi - 90°$, $\varphi - \theta + 180°$, $\varphi - 270°$, $\varphi + \theta + 60°$.

3. Change to circular measure each of the following angles: 88° 2′, 271° 53′, —18° 7′, 60°, 390°.

4. The circular measures of certain angles are as follows: 2., 16.2, 6., $-\frac{\pi}{8}$; find in each case the corresponding number of degrees.

5. Find the lengths of the following arcs in a circle whose radius is unity: 60°, 30°, 150°, 330°, 18°, 268°, 135°.

6. Find the lengths of the above arcs in a circle whose radius is 6.

7. Taking the earth's equatorial radius to be 3963 miles, find the length of an arc of 1° on the equator.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

§ **22.** A right triangle can always be formed by letting fall a perpendicular from *any* point in the terminal line of a given angle upon the initial line, produced if necessary. This triangle we will call a triangle of reference for the angle; and, in applying the rules of § 20 to determine the signs of its sides, we shall always regard the hypothenuse and base as measured *from* the vertex of the given angle, and the perpendicular as measured *from* the initial line.

In each of the figures 22, 23, 24, and 25, for example,

FIG. 22.

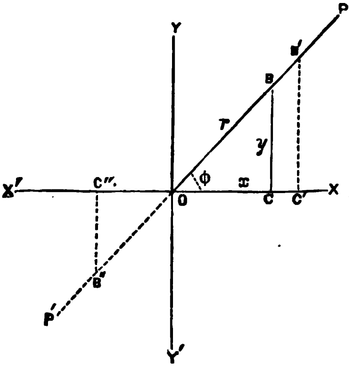


FIG. 23.

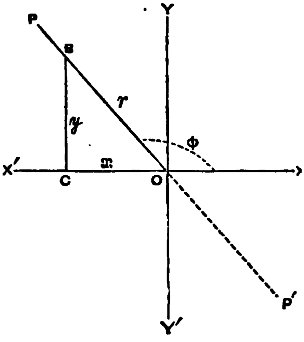


FIG. 24.

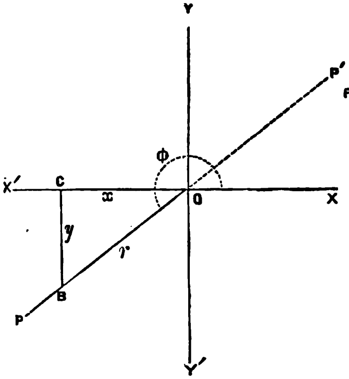
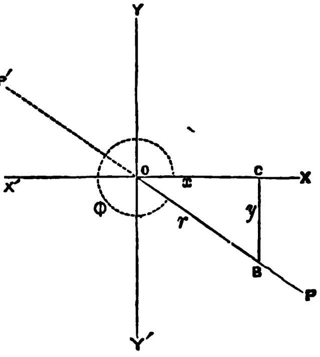


FIG. 25.



where φ (XOP) is, in each case, the given angle, we can form the right triangle OCB, as stated above. In order to form this triangle in figures 23 and 24, we must produce the line OX in the direction OX′.

In the above figures, the letters x, y, and r represent the base (OC), the perpendicular (CB), and the hypothenuse (OB) respectively, both in direction and length. Therefore x is positive in figures 22 and 25, and negative in figures 23 and 24; y is positive in figures 22 and 23, and negative in figures 24 and 25; r is positive in each figure. But r might be made negative in each figure. Thus, in Fig. 22, we may take $x = OC''$, $y = C''B''$, $r = OB''$, making x, y, and r all negative.

§ **23.** The *Trigonometric Functions* of an angle are the six ratios between the three sides of a triangle formed as above, taking into account the signs as well as the lengths of these sides.

These ratios are always the same for the same angle, both in numerical value and in sign; for if, as in figure 22, we let fall perpendiculars from different points of the terminal line, as B, B′, and B″, we have the right triangles OCB, OC′B′, and OC″B″, which evidently have their acute angles at O equal. These triangles are therefore (*v.* § 21, *d*) similar, and their homologous sides are proportional; i.e., the six ratios of the sides of OCB are, in numerical value, respectively equal to the six ratios of the sides of OC′B′, or OC″B″. They are equal in sign as well; for, in comparing any two of the triangles, the sides of the first either have the same directions, and therefore the same signs, as the corresponding sides of the second, as in OCB and OC′B′; or, as in OCB and OC″B″, each side of the second has a direction, and therefore a sign, opposite to that of the corresponding side of the first; and in each case the signs of the *ratios* are obviously the same.

These ratios are, in general, different for different angles; for two right triangles, *not* having an acute angle of the one *equal* to an acute angle of the other, are *not similar*, and their homologous sides are *not proportional.*

Thus we see that the values of the trigonometric ratios of an angle *depend*, and depend *only*, upon the value of the angle; for this reason they are called the *trigonometric functions** of the angle.

As soon as we have given names to these ratios, we shall be able to introduce into the same formula two entirely different kinds of quantities, — angles and straight lines. We shall then have made a very important step in the treatment of our subject.

---

* When one quantity depends upon another for its value, so that a change in the second necessitates a change in the first, the first is called a *function* of the second.

§ **24.** To each of the six ratios mentioned above, a name has been given as follows: In each of the figures 22, 23, 24, and 25, —

			is called the	abbreviated into
$\frac{CB}{OB}$	$\frac{y}{r}$	$\frac{\text{perpendicular}}{\text{hypothenuse}}$	sine of φ	sin φ
$\frac{OC}{OB}$	$\frac{x}{r}$	$\frac{\text{base}}{\text{hypothenuse}}$	cosine of φ	cos φ
$\frac{CB}{OC}$	$\frac{y}{x}$	$\frac{\text{perpendicular}}{\text{base}}$	tangent of φ	tan φ
$\frac{OC}{CB}$	$\frac{x}{y}$	$\frac{\text{base}}{\text{perpendicular}}$	cotangent of φ	ctn φ
$\frac{OB}{OC}$	$\frac{r}{x}$	$\frac{\text{hypothenuse}}{\text{base}}$	secant of φ	sec φ
$\frac{OB}{CB}$	$\frac{r}{y}$	$\frac{\text{hypothenuse}}{\text{perpendicular}}$	cosecant of φ	csc φ

In addition to the above

$1 - \cos\varphi$	versed sine of φ	vers φ

§ **25.** Bearing in mind what has been said about the signs of x, y, and r, in § 22, we easily deduce the following table for the signs of the functions of angles in each quadrant: —

φ	sin	cos	tan	ctn	sec	csc
1st quad	+	+	+	+	+	+
2d quad	+	−	−	−	−	+
3d quad	−	−	+	+	−	−
4th quad	−	+	−	−	+	−

Since the hypothenuse of a right triangle is the greatest of the three sides, the first two of our six ratios never exceed 1 in numerical value, and the last two never fall below 1 in numerical value. Hence *the sine and cosine vary only from* — 1 *to* + 1 ; *the secant and cosecant vary only from* + 1 *to* + ∞ *and from* — 1 *to* — ∞ ; *while the tangent and cotangent may have any value from* — ∞ *to* + ∞ .

§ 26. By § 24 we have, —

$$\left.\begin{array}{lcl}
\sin\varphi = \frac{y}{r}, & & \sin\varphi = \frac{1}{\csc\varphi}, \\
 & \therefore \sin\varphi \times \csc\varphi = 1 \;\therefore & \\
\csc\varphi = \frac{r}{y}, & & \csc\varphi = \frac{1}{\sin\varphi}, \\
\cos\varphi = \frac{x}{r}, & & \cos\varphi = \frac{1}{\sec\varphi}, \\
 & \therefore \cos\varphi \times \sec\varphi = 1 \;\therefore & \\
\sec\varphi = \frac{r}{x}, & & \sec\varphi = \frac{1}{\cos\varphi}, \\
\tan\varphi = \frac{y}{x}, & & \tan\varphi = \frac{1}{\text{ctn}\,\varphi}, \\
 & \therefore \tan\varphi \times \text{ctn}\,\varphi = 1 \;\therefore & \\
\text{ctn}\,\varphi = \frac{x}{y}, & & \text{ctn}\,\varphi = \frac{1}{\tan\varphi},
\end{array}\right\} \quad [1]$$

Therefore the *sine*, *cosine*, and *tangent* are reciprocals of the *cosecant*, *secant*, and *cotangent* respectively.

§ 27. $$\frac{\sin\varphi}{\cos\varphi} = \frac{y}{r} \div \frac{x}{r} = \frac{y}{x} = \tan\varphi, \;\therefore \frac{\sin\varphi}{\cos\varphi} = \tan\varphi. \quad [2]$$

$$\frac{\cos\varphi}{\sin\varphi} = \frac{x}{r} \div \frac{y}{r} = \frac{x}{y} = \text{ctn}\,\varphi, \;\therefore \frac{\cos\varphi}{\sin\varphi} = \text{ctn}\,\varphi. \quad [3]$$

Any two of the quantities appearing in either [2] or [3] being given, it is evident that the third can be found.

§ **28**. *Relations between sine and cosine, tangent and secant, cotangent and cosecant.*

By § 21 *e*, we have (Figs. 22–25) $x^2 + y^2 = r^2$, a formula which holds for all the figures since the *squares* of x, y, and r are positive in all cases; and from this we get: —

$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1, \therefore \sin^2\varphi + \cos^2\varphi = 1, \qquad [4]$$

$$\frac{r^2}{x^2} = 1 + \frac{y^2}{x^2}, \therefore \sec^2\varphi = 1 + \tan^2\varphi, \qquad [5]$$

$$\frac{r^2}{y^2} = 1 + \frac{x^2}{y^2}, \therefore \csc^2\varphi = 1 + \operatorname{ctn}^2\varphi, \qquad [6]$$

The above formulas enable us to solve the following problems: —

To find cos φ when sin φ is given, and the converse.
sec φ " tan φ " " "
csc φ " ctn φ " " "

In order, however, to solve these problems completely, it is also necessary to know to what quadrant φ belongs. When, for example, cos φ is given, we have from [4] $\sin\varphi = \pm\sqrt{1 - \cos^2\varphi}$, which does not, on account of the double sign of the radical, completely determine sin φ; but, when we also know to what quadrant φ belongs, we know by § 25 the sign of its sine, and can therefore completely determine sin φ. It should be noted that formulas [1]–[6] enable us, when any one function is given, to find all the others. The student is advised, before reading farther, to do examples 1–9 at the end of this chapter.

§ **29.** *To represent the trigonometric functions by straight lines.*

In order that the trigonometric functions, or ratios, may be represented by straight lines, the triangle of reference must be so taken for each ratio that the denominator of the ratio shall be unity.

In Figs. 26–29, let φ, or the angle XOP, be an angle of the 1st, 2d, 3d, and 4th quadrants respectively. From O as a centre, with a radius unity, describe a circle.

FIG. 26.

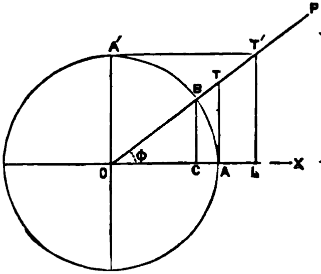


FIG. 27.

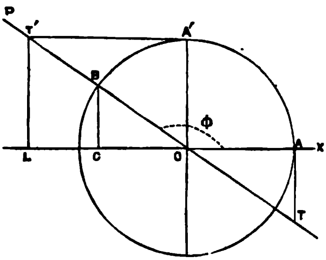


FIG. 28.

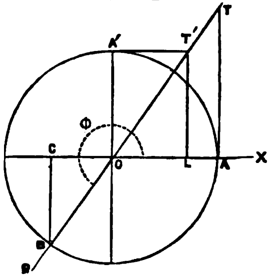
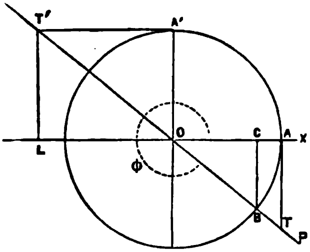


FIG. 29.



$\sin\varphi = \frac{\text{perp}}{\text{hyp}}$, $\cos\varphi = \frac{\text{base}}{\text{hyp}}$. Since the hypothenuse must be unity, we will, in each figure, construct the triangle of reference OCB, in which the hypothenuse OB is equal to the radius, or unity.

$$\therefore \sin \varphi = \frac{\text{CB}}{1} = \text{CB}, \cos \varphi = \frac{\text{OC}}{1} = \text{OC}.$$

$\tan \varphi = \frac{\text{perp}}{\text{base}}$, $\sec \varphi = \frac{\text{hyp}}{\text{base}}$. By drawing a geometric tangent at A, we have the triangle of reference OAT, in which the base OA is unity.

$$\therefore \tan \varphi = \frac{\text{AT}}{1} = \text{AT}, \sec \varphi = \frac{\text{OT}}{1} = \text{OT}.$$

$\text{ctn}\, \varphi = \frac{\text{base}}{\text{perp}}$, $\csc \varphi = \frac{\text{hyp}}{\text{perp}}$. Through a point + 90 ° from A, draw a geometric tangent which will cut OB produced in some point as T′; from T′ let fall a perpendicular upon X′X. The triangle of reference OLT′ thus formed will be a triangle in which the perpendicular $\text{LT}' = \text{OA}' = 1$, and the base $\text{OL} = \text{A}'\text{T}'$.

$$\therefore \text{ctn}\, \varphi = \frac{\text{OL}}{1} = \text{OL} = \text{A}'\text{T}', \csc \varphi = \frac{\text{OT}'}{1} = \text{OT}'.$$

$$\text{versed} \sin \varphi = 1 - \cos \varphi = \text{OA} - \text{OC} = \text{CA}$$

From § 20 and § 22 we know that: —

In Fig. 26, —

OC, CB, AT, OT, A′T′, OT′, and CA are positive.

In Fig. 27, —

CB, OT′, and CA are positive,
and OC, AT, OT, and A′T′ are negative.

In Fig. 28, —

AT, A′T′, and CA are positive,
and CB, OC, OT, and OT′ are negative.

In Fig. 29, —

OC, OT, and CA are positive,
and CB, AT, A′T′, and OT′ are negative.

Putting the above results in words, we have the following corollaries to our original definitions (*v.* § 24): —

In a circle whose radius is unity, the *trigonometric functions* of an angle or arc may be *represented* by the following lines: —

The *sine*, by the line which measures the perpendicular distance of the end of the arc *from** the initial line.

The *cosine*, by the line drawn *from* the centre of the circle *to* the foot of the *sine.*

The *tangent*, by that part of the geometric tangent at the beginning of the arc which is drawn *from* the point of tangency *to* the terminal line.

The *cotangent*, by that part of the geometric tangent at a point + 90° from the beginning of the arc which is drawn *from* the point of tangency *to* the terminal line.

The *secant*, by the line drawn *from* the centre of the circle *to* the end of the tangent.

The *cosecant*, by the line drawn *from* the centre of the circle *to* the end of the cotangent.

The *versed sine*, by the line drawn *from* the foot of the sine to the beginning of the arc.

What has been said in this section should not lead the student to regard the trigonometric functions as absolute lines. The trigonometric functions are *not* the lines themselves, but are merely the *numbers* which denote the ratios of the lines to the arbitrary unit which we choose as a radius.

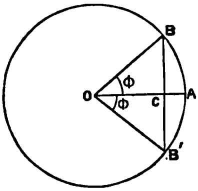
---

* The word "from" indicates the direction, and therefore the sign of the line.

§ **30.** *In any circle the radius of which is unity, the sine of any arc is one-half the chord of twice the arc.*

In Fig. 30 let the angle B′OB or the arc $B'B = 2\,\varphi$. Draw OA so as to bisect the angle or $2\,\varphi$; draw the chord B′B. Since OA bisects the angle B′OB, we know from geometry that it is perpendicular to and bisects the chord B′B. $\therefore\ CB = \frac{1}{2}\,B'B = \frac{1}{2}$ chord $2\,\varphi$.

FIG. 30.

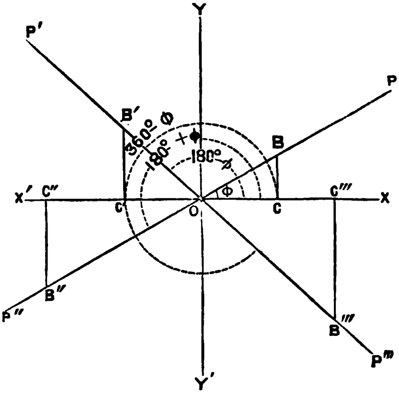


But by § 29, —

$$CB = \sin\varphi \ \therefore\ \sin\varphi = \tfrac{1}{2}\text{ chord } 2\,\varphi \qquad [7]$$

§ **31*** *Relations between the functions of* $\varphi \pm k\,360°$, $180° - \varphi$, $180° + \varphi$, $360° - \varphi$, $-\varphi$, *and the functions of* φ.

FIG. 31.



Let us take any angle of the 1st quadrant, as XOP (Fig. 31); let the acute value of this angle be denoted by φ. XOP then denotes either the angle φ or the angle $\varphi \pm k\,360°$ (*v.* § 18).

---

* A treatment of the subject of §§ 31, 32, and 33, covering explicitly all possible values of φ, is given in appendix I., which the teacher may at his discretion substitute for these sections.

Construct the angles :—

$$XOP' = 180° - \varphi, \quad XOP'' = 180° + \varphi,$$
$$XOP''' = 360° - \varphi \text{ or } = -\varphi.$$

In accordance with this construction, the acute angles XOP or φ, $P'OX'$, $X'OP''$, and $P'''OX$ are all equal to one another.

Construct the triangles of reference OCB, $OC'B'$, $OC''B''$, and $OC'''B'''$; each of these triangles having an acute angle equal to φ, they are similar (*v.* § 21, *d*), and their homologous sides are proportional; moreover, the perpendiculars CB, $C'B'$, $C''B''$, and $C'''B'''$, being opposite the equal angles, are homologous. Therefore the corresponding trigonometric functions of the angles φ, $\varphi \pm k\ 360°$, $180° - \varphi$, $180° + \varphi$, $360° - \varphi$, and $-\varphi$, are numerically equal; but, paying proper attention to the algebraic signs of these functions (*v.* § 25), we arrive at the following results :—

$$\left.\begin{aligned}\sin(\varphi \pm k\ 360°) &= \sin\varphi\\ \cos(\varphi \pm k\ 360°) &= \cos\varphi\end{aligned}\right\} \qquad [8]$$

$$\left.\begin{aligned}\frac{C'B'}{OB'} &= \frac{CB}{OB} \therefore \sin(180° - \varphi) = \sin\varphi\\ \frac{OC'}{OB'} &= -\frac{OC}{OB} \therefore \cos(180° - \varphi) = -\cos\varphi\end{aligned}\right\} \qquad [9]$$

$$\left.\begin{aligned}\frac{C''B''}{OB''} &= -\frac{CB}{OB} \therefore \sin(180° + \varphi) = -\sin\varphi\\ \frac{OC''}{OB''} &= -\frac{OC}{OB} \therefore \cos(180° + \varphi) = -\cos\varphi\end{aligned}\right\} \qquad [10]$$

$$\left.\begin{aligned}\frac{C'''B'''}{OB'''} &= -\frac{CB}{OB} \therefore \sin(360° - \varphi) = -\sin\varphi\\ &\qquad\qquad\qquad\qquad\quad = \sin(-\varphi)\\ \frac{OC'''}{OB'''} &= \frac{OC}{OB} \therefore \cos(360° - \varphi) = \cos\varphi\\ &\qquad\qquad\qquad\qquad\quad = \cos(-\varphi)\end{aligned}\right\} \qquad [11]$$

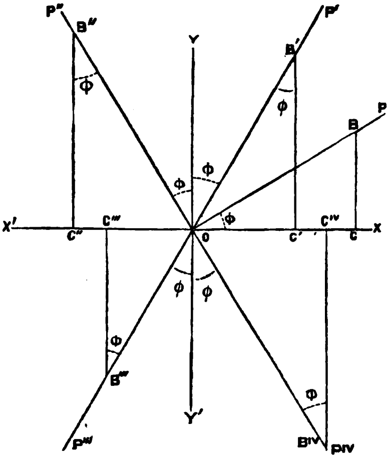
In like manner the other functions of these angles may be found; they may also be found from the above by [1], [2], and [3].

§ **32.** *Relations between the functions of* $90° + \varphi$, $90° - \varphi$, $270° + \varphi$, $270° - \varphi$, *and the functions of* φ.

Let us take any angle of the 1st quadrant, as XOP (Fig. 32), which we will call φ. Construct the angles,—

$$XOP' = 90° - \varphi,\ XOP'' = 90° + \varphi,$$
$$XOP''' = 270° - \varphi,\ XOP^{iv} = 270° + \varphi.$$

FIG. 32.



Construct, with respect to these angles, the triangles of reference OCB, $OC'B'$, $OC''B''$, $OC'''B'''$, and $OC^{iv}B^{iv}$. In accordance with this construction, the angles marked φ in the figure are all equal.

The triangles OCB, $OC'B'$, $OC''B''$, $OC'''B'''$, and $OC^{iv}B^{iv}$, having each an acute angle equal to φ, are all similar (*v.* § 21, *d*), and their homologous sides are proportional. But in this case the lines OC', OC'', OC''', and OC^{iv} of the triangles $OC'B'$, $OC''B''$, $OC'''B'''$, and $OC^{iv}B^{iv}$, being opposite the angle φ, are homologous with CB of the triangle OCB; and $C'B'$, $C''B''$, $C'''B'''$, and $C^{iv}B^{iv}$, are homologous with OC. Therefore, paying proper attention to the algebraic signs, we get the following results:—

$$\left.\begin{aligned}\frac{C'B'}{OB'} &= \frac{OC}{OB} \therefore \sin(\ 90^\circ - \varphi) = \cos\varphi\\ \frac{OC'}{OB'} &= \frac{CB}{OB} \therefore \cos(\ 90^\circ - \varphi) = \sin\varphi\end{aligned}\right\} \quad [12]$$

$$\left.\begin{aligned}\frac{C''B''}{OB''} &= \frac{OC}{OB} \therefore \sin(\ 90^\circ + \varphi) = \cos\varphi\\ \frac{OC''}{OB''} &= -\frac{CB}{OB} \therefore \cos(\ 90^\circ + \varphi) = -\sin\varphi\end{aligned}\right\} \quad [13]$$

$$\left.\begin{aligned}\frac{C'''B'''}{OB'''} &= -\frac{OC}{OB} \therefore \sin(270^\circ - \varphi) = -\cos\varphi\\ \frac{OC'''}{OB'''} &= -\frac{CB}{OB} \therefore \cos(270^\circ - \varphi) = -\sin\varphi\end{aligned}\right\} \quad [14]$$

$$\left.\begin{aligned}\frac{C^{iv}B^{iv}}{OB^{iv}} &= -\frac{OC}{OB} \therefore \sin(270^\circ + \varphi) = -\cos\varphi\\ \frac{OC^{iv}}{OB^{iv}} &= \frac{CB}{OB} \therefore \cos(270^\circ + \varphi) = \sin\varphi\end{aligned}\right\} \quad [15]$$

§ **33.** From [12], by [1], [2], and [3], we readily obtain the following: —

$$\left.\begin{aligned}\csc(90^\circ - \varphi) &= \sec\varphi\\ \sec(90^\circ - \varphi) &= \csc\varphi\\ \tan(90^\circ - \varphi) &= \operatorname{ctn}\varphi\\ \operatorname{ctn}(90^\circ - \varphi) &= \tan\varphi\end{aligned}\right\} \quad [12_1]$$

Now, φ and $90^\circ - \varphi$ are complements of each other: therefore from [12] and [12_1] we see that the *sine*, *tangent*, and *secant* of any angle are respectively the *cosine*, *cotangent*, and *cosecant* of its complement.

"Co", then, is merely an abbreviation for the word "complement's;" [12] and [12_1] may therefore be expressed in words as follows: —

The *cosine* of any angle is its *complement's sine.*

The *cosecant* of any angle is its *complement's secant.*

The *cotangent* of any angle is its *complement's tangent.*

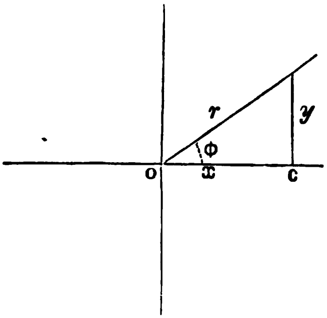
Formulas 8–15 are easily remembered if we note that when φ is coupled with 0°, 180°, or k 360°, the functions of the angles thus formed, and those of φ, are *numerically the same;* while, when φ is coupled with 90° or 270°, the functions of the angles thus formed, and those of φ, are *numerically complementary.*

The *algebraic sign* is determined in each case by supposing φ to be acute, and then noticing to what quadrant the resulting angle belongs.

It will be readily seen, by constructing proper figures, that formulas 8–15 hold for all values of φ.

§ 34. *To find the functions of 0° and 360°.*

FIG. 33.



Suppose φ (Fig. 33) to decrease until it becomes 0; then y will become 0, and r will coincide with x.

$$\therefore \sin 0^\circ = \frac{y}{r} = \frac{0}{r} = 0,$$

$$\cos 0^\circ = \frac{x}{r} = 1,$$

$$\tan 0^\circ = \frac{y}{x} = \frac{0}{x} = 0, \qquad \sec 0^\circ = \frac{r}{x} = 1,$$

$$\text{ctn } 0^\circ = \frac{x}{y} = \frac{x}{0} = \infty, \qquad \csc 0^\circ = \frac{r}{y} = \frac{r}{0} = \infty.$$

When φ is equal to 360° we shall also have $y = 0$, and $r = x$; therefore the functions of 360° are the same as those of 0°.

§ **35.** *To find the functions of 90°.*

Suppose (Fig. 33) φ to increase until it becomes equal to 90°; then x will become 0, and y will coincide with r.

$$\therefore \sin 90^\circ = \frac{y}{r} = 1, \qquad \csc 90^\circ = \frac{r}{y} = 1,$$

$$\cos 90^\circ = \frac{x}{r} = \frac{0}{r} = 0, \qquad \sec 90^\circ = \frac{r}{x} = \frac{r}{0} = \infty,$$

$$\tan 90^\circ = \frac{y}{x} = \frac{y}{0} = \infty \qquad \text{ctn } 90^\circ = \frac{x}{y} = \frac{0}{y} = 0.$$

These results can also be found by putting $\varphi = 0$ in [12] and [12_1].

§ **36.** *To find the functions of 180°.*

Suppose φ (Fig. 33) to increase until it becomes 180°: then y will become 0, and r will become *numerically* equal to x, but opposite in *sign*. (*v.* § 20, *d.*)

$$\therefore \sin 180^\circ = \frac{y}{r} = \frac{0}{r} = 0, \quad \csc 180^\circ = \frac{r}{y} = \frac{r}{0} = \infty,$$

$$\cos 180^\circ = \frac{x}{r} = -1, \quad \tan 180^\circ = \frac{y}{x} = \frac{0}{x} = 0,$$

$$\sec 180^\circ = \frac{r}{x} = -1, \quad \text{ctn } 180^\circ = \frac{x}{y} = \frac{x}{0} = \infty.$$

Sin 180° and cos 180° may also be found by putting $\varphi = 0$ in [9], and from these the other functions by [1], [2], and [3].

§ **37.** *To find the functions of 270°.*

Suppose φ (Fig. 33) to increase until it becomes 270°: then x will become 0, and r will become *numerically* equal to y, but opposite in *sign*. (*v.* § 20, *d.*)

$$\therefore \sin 270^\circ = \frac{y}{r} = -1 \qquad \csc 270^\circ = \frac{r}{y} = -1$$

$$\cos 270^\circ = \frac{x}{r} = \frac{0}{r} = 0 \qquad \sec 270^\circ = \frac{r}{0} = \infty$$

$$\tan 270^\circ = \frac{y}{x} = \frac{y}{0} = \infty \qquad \text{ctn}\, 270^\circ = \frac{x}{y} = \frac{0}{y} = 0$$

§ **38.** *To find the functions of 30° and 60°.*

In formula 7 let $\varphi = 30^\circ$; this formula will then become $\sin 30^\circ = \frac{1}{2}$ chord 60°. But by § 21, *i*, chord 60° = radius = 1. $\therefore \sin 30^\circ = \frac{1}{2}$. $\therefore$ from [4] we have, —

$$\cos 30^\circ = +\sqrt{1 - \sin^2 30} = +\sqrt{1 - \tfrac{1}{4}} = +\sqrt{\tfrac{3}{4}}$$
$$= +\tfrac{1}{2}\sqrt{3};^*$$

and from [2], —

$$\tan 30^\circ = \frac{\sin 30}{\cos 30} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}};$$

from [3], —

$$\text{ctn}\, 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3};$$

from [1], —

$$\sec 30^\circ = \frac{1}{\cos 30} = \frac{2}{\sqrt{3}}, \ \csc 30^\circ = \frac{1}{\sin 30} = 2.$$

---

* Why + ? *v.* remark at end of § 28.

We have then by [12] and [12_1] since

$$60° = 90° - 30°;$$

$$\sin 60 = \cos 30° = \tfrac{1}{2}\sqrt{3}, \quad \cos 60° = \sin 30° = \tfrac{1}{2},$$

$$\tan 60° = \text{ctn}\, 30° = \sqrt{3}, \quad \text{ctn}\, 60° = \tan 30° = \frac{1}{\sqrt{3}},$$

$$\sec 60° = \csc 30° = 2, \quad \csc 60° = \sec 30° = \frac{2}{\sqrt{3}}.$$

§ 39. *To find the functions of 45°.*

45° is the complement of 45°. Therefore by [12] and [12_1], —

$$\sin 45 = \cos 45, \quad \tan 45° = \text{ctn}\, 45°, \quad \sec 45° = \csc 45°.$$

Whence by [4], —

$$\sin^2 45° + \cos^2 45° = 1 = 2 \sin^2 45° = 2 \cos^2 45°.$$

$$\therefore \sin 45° = \cos 45° = +\sqrt{\frac{1}{2}},*$$

$$\tan 45° = \text{ctn}\, 45° = \frac{\sin 45°}{\cos 45°} = 1,$$

$$\sec 45° = \csc 45° = \frac{1}{\sin 45°} = \sqrt{2}.$$

§ 40. *To find the functions of 120°.*

$$120° = 180° - 60°;$$

∴ by [9], —

$$\sin 120° = \sin 60° = \tfrac{1}{2}\sqrt{3},$$

$$\cos 120° = -\cos 60° = -\tfrac{1}{2},$$

---

* Why +? (*v.* remark at end of § 28.)

$$\tan 120^\circ = \frac{\sin 120^\circ}{\cos 120^\circ} = \frac{\frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = -\sqrt{3},$$

$$\text{ctn } 120^\circ = \frac{1}{\tan 120^\circ} = -\frac{1}{\sqrt{3}},$$

$$\sec 120^\circ = \frac{1}{\cos 120^\circ} = \frac{1}{-\frac{1}{2}} = -2,$$

$$\csc 120^\circ = \frac{1}{\sin 120^\circ} = \frac{1}{\frac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

§ **41.** *To find the functions of 150°.*

$$150^\circ = 180^\circ - 30^\circ$$

∴ by [9], —

$$\sin 150^\circ = \sin 30^\circ = \tfrac{1}{2}$$

$$\cos 150^\circ = -\cos 30^\circ = -\tfrac{1}{2}\sqrt{3},$$

$$\tan 150^\circ = \frac{\sin 150^\circ}{\cos 150^\circ} = \frac{\frac{1}{2}}{-\frac{1}{2}\sqrt{3}} = -\frac{1}{\sqrt{3}},$$

$$\text{ctn } 150^\circ = \frac{1}{\tan 150^\circ} = -\sqrt{3},$$

$$\sec 150^\circ = \frac{1}{\cos 150^\circ} = -\frac{2}{\sqrt{3}},$$

$$\csc 150^\circ = \frac{1}{\sin 150^\circ} = \frac{1}{\frac{1}{2}} = 2.$$

§ **42.** *To find the functions of 135°.*

$$135^\circ = 180^\circ - 45^\circ;$$

$\therefore$ by [9], —

$$\sin 135° = \sin 45° = \sqrt{\tfrac{1}{2}}\ ,$$

$$\cos 135° = -\cos 45° = -\sqrt{\tfrac{1}{2}},$$

$$\tan 135° = \frac{\sin 135°}{\cos 135°} = \frac{\sqrt{\tfrac{1}{2}}}{-\sqrt{\tfrac{1}{2}}} = -1,$$

$$\text{ctn}\, 135° = \frac{1}{\tan 135°} = \frac{1}{-1} = -1,$$

$$\sec 135° = \frac{1}{\cos 135°} = \frac{1}{-\sqrt{\tfrac{1}{2}}} = -\sqrt{2},$$

$$\csc 135° = \frac{1}{\sin 135°} = \frac{1}{\sqrt{\tfrac{1}{2}}} = \sqrt{2}.$$

§ **43.** Collecting the results of the last nine sections for easy reference, we have: —

Angle	sin	cos	tan	ctn	sec	csc
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\sqrt{\frac{1}{3}}$	$\sqrt{3}$	$2\sqrt{\frac{1}{3}}$	2
45°	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\sqrt{\frac{1}{3}}$	2	$2\sqrt{\frac{1}{3}}$
90°	1	0	∞	0	∞	1
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	-2	$2\sqrt{\frac{1}{3}}$
135°	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{3}$	$-2\sqrt{\frac{1}{3}}$	2
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1
360°	0	1	0	∞	1	∞

[16]

§ 44. *Increase and decrease of the functions.*

We have shown in § 30 that, in the circle of which the radius is unity, —

$$\sin\varphi = \tfrac{1}{2}\text{ chord } 2\varphi.$$

Now, we know by geometry that the increase of an arc less than a semi-circumference is accompanied by the increase of its chord. Hence, if $\varphi < 90°$, that is, if $2\varphi < 180°$, $\sin\varphi$ increases with the increase of φ. Since, when $\varphi < 90°$, $\cos\varphi = +\sqrt{1 - \sin^2\varphi}$, $\cos\varphi$ decreases with the increase of $\sin\varphi$, and therefore decreases with the increase of φ. Again, $\tan\varphi = \frac{\sin\varphi}{\cos\varphi}$, of which the numerator increases and the denominator decreases with the increase of φ. Either of these changes causes the increase of the fraction. Hence, if $\varphi < 90°$, $\tan\varphi$ increases with the increase of φ. The reciprocal of a quantity decreases with the increase of the quantity, and *vice versa.* Hence we see that *the sine, tangent, and secant of an acute angle increase with the increase of the angle, while the cosine, cotangent, and cosecant decrease.*

Now, if φ is acute, $180° - \varphi$, $180° + \varphi$, and $360° - \varphi$, are general expressions for angles in the second, third, and fourth quadrants respectively; and $180° + \varphi$ increases with the increase of φ, while $180° - \varphi$ and $360° - \varphi$ decrease with the increase of φ. But the functions of these angles are *numerically* equal to those of φ, by § 31. Hence, in the third quadrant, the rule for the increase and decrease of the numerical values of the functions is the same as in the first; but in the second and fourth quadrants the rule is opposite to the first.

The *algebraic* value of a negative quantity (that is, the complete value, taking account of the sign) increases with the decrease of its numerical value, and *vice versa.* Regarding, then, the table of signs in § 25, we obtain the following results: —

	sin.	cos.	tan.	ctn.	sec.	csc.
0°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$
1st qu.	inc.	dec.	inc.	dec.	inc.	dec.
90°	$+1$	± 0	$\pm\infty$	± 0	$\pm\infty$	$+1$
2d qu.	dec.	dec.	inc.	dec.	inc.	inc.
180°	± 0	-1	∓ 0	$\mp\infty$	-1	$\pm\infty$
3d qu.	dec.	inc.	inc.	dec.	dec.	inc.
270°	-1	∓ 0	$\pm\infty$	± 0	$\mp\infty$	-1
4th qu.	inc.	inc.	inc.	dec.	dec.	dec.
360°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$

[17]

In this table, the sign before ∞ or 0 shows whether the function is passing from $+$ to $-$ or from $-$ to $+$, with the increase of the angle.

§ **45**. *Inverse or anti functions.*

In $x = \sin\varphi$, x is expressed as a function of φ. In order to express φ as a function of x, the following notation has been adopted: —

$$\varphi = \sin^{-1} x.$$

This equation denotes that φ is that angle whose sine is x.

In like manner, if $y = \cos\varphi$, $z = \tan\varphi$, and $w = \sec\varphi$, we can write: —

$$\varphi = \cos^{-1} y, \qquad \varphi = \tan^{-1} z, \qquad \varphi = \sec^{-1} w.$$

Sin$^{-1} x$, cos$^{-1} y$, tan$^{-1} z$, &c., are called the *inverse* or *anti* trigonometric functions, and are read antisine of x, anticosine of y, &c.; they are sometimes written arc sin x, arc cos y, arc tan z, &c., since they denote the *arc*, or angle, whose sine is x, whose cosine is y, &c.

In like manner we can denote the inverse or anti-logarithmic functions. Thus, if $\log u = x$, then $u = \log^{-1} x$.

The inverse of "log sin" may be written, $(\log\sin)^{-1}$. Thus, if $\log\sin\varphi = 9.8421$, then $\varphi = (\log\sin)^{-1}\, 9.8421$.

Examples: —

$\sin 45° = \sqrt{\tfrac{1}{2}}$, $\tan 30° = \sqrt{\tfrac{1}{3}}$, $\csc 270° = -1$;

therefore,

$45° = \sin^{-1}\sqrt{\tfrac{1}{2}}$, $30° = \tan^{-1}\sqrt{\tfrac{1}{3}}$, $270° = \csc^{-1}(-1)$.

Any relation which can be shown to exist between the trigonometric functions can be expressed by means of the inverse notation. If, for example, —

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{\sin}{\cos}\,\varphi = x,$$

then $$\varphi = \tan^{-1} x = \left(\frac{\sin}{\cos}\right)^{-1} x.$$

EXAMPLES.

1. What signs belong to the different functions of each of the following angles?

289°, 368°, 510°, 640°, 1178°, —80°, —188°, —1722°.

2. In what quadrants must an angle be taken whose secant $= +\frac{5}{4}$?

In what quadrants must an angle be taken whose tangent $= -\frac{4}{5}$?

In what quadrants must an angle be taken whose cosecant $= -2$? 3, 4

3. In what quadrants must A, A_1, &c., be taken in each of the following cases?

1, 2 $\sin A = +0.809$ $\cos A_1 = -0.599$ 2, 3
2, 4 $\tan A_2 = -\sqrt{\tfrac{1}{3}}$ $\text{ctn}\, A_3 = +0.466$ 1, 3
2, 3 $\sec A_4 = -\sqrt{2}$ $\csc A_5 = +3.86.$ 1, 2

4. Which of the trigonometric functions of an angle are never numerically less than unity? which never numerically greater than unity? which sometimes greater and sometimes less than unity?

5. Given $\sin A = \frac{1}{2}\sqrt{3}$, and $\cos A = \frac{1}{2}$; find all the other functions of A.

6. Given $\cos A = -\sqrt{\tfrac{1}{2}}$; find all the other functions of A when A is an angle of the 2d quadrant.

7. Given $\cos A = \dfrac{1}{2\sqrt{2}}$; find all the other functions of A when A is an angle of the 4th quadrant.

8. Given $\tan A = \sqrt{3}$; find all the other functions of A when A is an angle of the 3d quadrant.

9. Given $\text{ctn}\, A = \frac{3}{4}$; find all the possible values of other functions of A.

10. Prove $\sec^2 A \cdot \csc^2 A = \sec^2 A + \csc^2 A$.

11. If $\csc A = \sec B$, what is the relation between A and B? (*v.* § 33.)

12. Draw lines in and about a circle whose radius is unity, which represent the functions of an angle A of the 2d quadrant when $\cos A = -\frac{1}{2}$; also when A is an angle of the 4th quadrant, and $\cos A = +\frac{1}{2}$.

13. Suppose A to be an angle of the 3d quadrant; draw lines as above when $\tan A = +1$.

14. Find by geometry all the functions of 45°. Given chord $36° = 0.618$, find $\sin 18°$.

15. Given $\sin A = \frac{1}{2}$, and $A < 90°$, what are the functions of the angles $-A$, $90° \pm A$, $180° \pm A$, $270° \pm A$, $360° \pm A$?

16. Given $\sin 30 = \frac{1}{2}$; what are the functions of the angles 3270°, 2580°, 750°, −150°, −60°, −120°?

17. Show that formulas 8–15 hold when φ is an angle of the 2d quadrant.

18. The same as the above when φ is an angle of the 3d quadrant.

19. Find all the functions of the following angles:

210°, 225°, 240°, 300°, 315°, −45°.

20. What values of A, A_1, A_2, and A_3, between 0° and 720°, will satisfy the following equations:—

$$\sin A = +\tfrac{1}{2} \qquad \cos A_1 = -\sqrt{\tfrac{1}{2}}$$

$$\tan A_2 = +\sqrt{3} \qquad \tan^2 A_3 + 4 \sin^2 A_3 = 0$$

21. What angles less than 360° have a cosine $= -\sqrt{\frac{1}{2}}$? what a secant $= -2$? what a tangent $= +\sqrt{3}$?

22. Which of the following angles have a cosine $= -\frac{1}{2}$? which a tangent $= -1$? which a secant $= +\sqrt{2}$? 45°, 120°, 225°, 240°, 315°, −240°, −315°, 600°.

23. What values of A between 0° and 720° will satisfy the equation $\sin^2 A + 5 \cos^2 A = 3$?

24. Given $\sin^2 A = m \cos A - n$; find $\cos A$.

25. Given $\sin A = m \sin B$, and $\tan A = n \tan B$; find $\sin A$ and $\cos B$.

26. Show what change is taking place in each of the functions of an angle when the angle is increasing from 0° to 90°, from 90° to 180°, from 180° to 270°, and from 270° to 360°.

27. Given $\sin \theta + \cos \theta = 1$; find θ.

28. Given $\sin^2\theta = \dfrac{3}{2\sec\theta}$; find θ.

29. Given $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$; find θ.

30. Prove: —

$$\sin^2\theta \,.\, \tan\theta + \cos^2\theta \,.\, \operatorname{ctn}\theta + 2\sin\theta\cos\theta = \tan\theta + \operatorname{ctn}\theta.$$

31. Given $\cos 288° = 0.309$; find $\operatorname{ctn} 162°$. (Carry the result to three places of decimals.)

32. Given $\tan 241° = 1.8$; find $\sin 299°$.

33. Given $A = \sin^{-1}\frac{3}{5}$, $B = \sin^{-1}\frac{4}{5}$;
Prove $A + B = 90°$.

34. Prove $\sin^{-1}x + \cos^{-1}x = 90°$.

35. Find the value of $\tan(\tan^{-1}x + \operatorname{ctn}^{-1}x)$.

36. $\operatorname{Cos} x = \dfrac{\cos A}{\sin C}$. $\operatorname{Cos}(90 - x) = \dfrac{\cos B}{\sin C}$.
Prove $\cos^2 A + \cos^2 B + \cos^2 C = 1$.

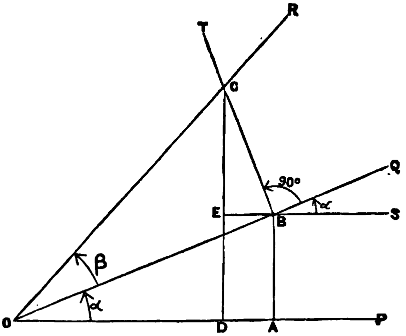
CHAPTER III.

FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ANGLES, AND FORMULAS DEDUCED FROM THEM.

§ **46.** *Given the sine and cosine of any two angles, to find the sine and cosine of their sum and of their difference.*

Let α and β denote the two angles, and let OP, OQ, and OR be so drawn in a plane that $\alpha = \text{POQ}$, $\beta = \text{QOR}$.

FIG. 34.



Then, by § 16, —

$$\alpha + \beta = \text{POQ} + \text{QOR} = \text{POR}.$$

The angles α, β, and $\alpha+\beta$, are all positive and acute in Fig. 34; but our solution is applicable to all possible cases (*v.* § 47).

From any point, C, of the line OR, drop perpendiculars on OP and OQ, meeting OP and OQ at D and B respectively. From B drop a perpendicular on OP, meeting OP at A. Then the triangles OAB, OBC, and ODC, are triangles of reference for α, β, and $\alpha+\beta$, respectively; and we have —

$$\sin\alpha = \frac{AB}{OB}; \therefore AB = \sin\alpha \,.\, OB; (a) \quad \sin\beta = \frac{BC}{OC}. (b)$$

$$\cos\alpha = \frac{OA}{OB}; \therefore OA = \cos\alpha \,.\, OB; (c) \quad \cos\beta = \frac{OB}{OC}. (d)$$

$$\sin(\alpha+\beta) = \frac{DC}{OC}; \quad \cos(\alpha+\beta) = \frac{OD}{OC}.$$

Through B draw BS parallel to OP, and meeting DC at E. Let S and Q be so taken that BS and BQ are positive; and let T be so taken on BC that QBT is a positive right angle. Then, by § 21 *h* and § 16, —

$$SBT = SBQ + QBT = 90° + \alpha;$$

and BEC is a triangle of reference for this angle. Hence, by [13], —

$$\sin(90° + \alpha) = \cos\alpha = \frac{EC}{BC}; \therefore EC = \cos\alpha \,.\, BC. (e)$$

$$\cos(90° + \alpha) = -\sin\alpha = \frac{BE}{BC}; \therefore BE = -\sin\alpha \,.\, BC. (f)$$

Again, by § 2 and § 3, *d*,—

$$DC = DE + EC = AB + EC; \quad OD = OA + AD = OA + BE.$$

Then we have —

$$\sin(\alpha + \beta) = \frac{DC}{OC} = \frac{AB}{OC} + \frac{EC}{OC},$$

$$\cos(\alpha + \beta) = \frac{OD}{OC} = \frac{OA}{OC} + \frac{BE}{OC};$$

or substituting from (*a*), (*e*), (*c*), and (*f*),

$$\sin(\alpha + \beta) = \sin\alpha \,.\, \frac{OB}{OC} + \cos\alpha \,.\, \frac{BC}{OC},$$

$$\cos(\alpha + \beta) = \cos\alpha \,.\, \frac{OB}{OC} - \sin\alpha \,.\, \frac{BC}{OC};$$

now, substituting from (*d*) and (*b*), we get —

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \qquad [18,\ a]$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \qquad [19,\ a]$$

Since the above solution is applicable to all values of α and β (*v.* § 47), we may replace β by $-\beta$.

But, by [11], —

$$\sin(-\beta) = -\sin\beta, \qquad \cos(-\beta) = \cos\beta.$$

Hence we have —

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta, \qquad [18,\ b]$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \qquad [19,\ b]$$

The four formulas proved above can be embodied in two formulas, as follows: —

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad [18]$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \qquad [19]$$

where we are to use either the upper signs throughout or the lower signs throughout.

These two formulas, which are constantly used in mathematics, should be committed to memory.

§ **47.** After the student has thoroughly mastered the above solution as applied to any one case of the problem, such as that of Fig. 34, he should carefully re-examine it, with the view to satisfying himself that it is equally valid for any possible case. To assist him in making this examination, we append Figs. 35–38; and the student is advised to draw figures which shall represent still other cases. But it is important to observe that the generality of the solution does not depend on a complete analysis of cases, but on its being stated in a form which can be seen at every point to admit of universal application.

FIG. 35.

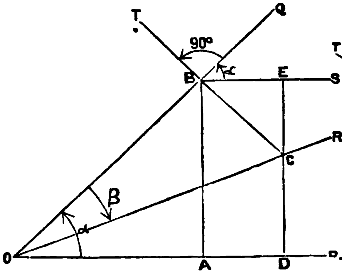


FIG. 36.

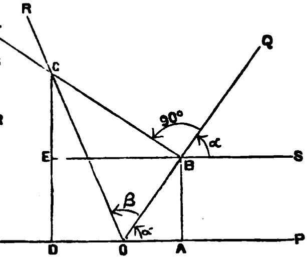


FIG. 37.

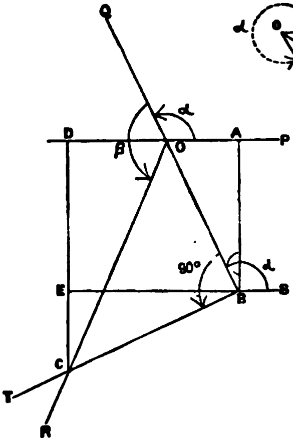
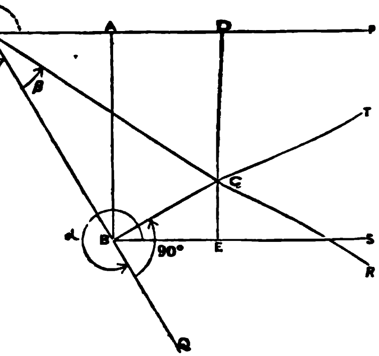


FIG. 38.



§ **48.** *Given the sine and cosine of the sum or difference of any two angles, to find the tangent and cotangent.*

By [2], [18], and [19], we have, —

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

Divide both numerator and denominator by cos α cos β, and we get, —

$$\tan(\alpha \pm \beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \pm \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \mp \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \qquad [20]$$

By [1] and [20] we have, —

$$\frac{1}{\tan(\alpha \pm \beta)} = \operatorname{ctn}(\alpha \pm \beta) = \frac{1 \mp \tan\alpha\tan\beta}{\tan\alpha \pm \tan\beta}$$

Divide both numerator and denominator by tan α tan β, and we get, —

$$\operatorname{ctn}(\alpha \pm \beta) = \frac{\operatorname{ctn}\alpha\operatorname{ctn}\beta \mp 1}{\operatorname{ctn}\beta \pm \operatorname{ctn}\alpha} \qquad [21]$$

§ **49.** The following formulas are sometimes found useful. Let the student prove them by the aid of formulas [2, 18 *a*, 18 *b*, 19 *a*, 19 *b*, and 4.]

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\frac{\operatorname{ctn}\beta+\operatorname{ctn}\alpha}{\operatorname{ctn}\beta-\operatorname{ctn}\alpha} \qquad [22]$$

$$\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\operatorname{ctn}\beta-\tan\alpha}{\operatorname{ctn}\beta+\tan\alpha} \qquad [23]$$

$$\left.\begin{aligned}\sin(\alpha+\beta)\sin(\alpha-\beta)&=\sin^2\alpha-\sin^2\beta\\&=\cos^2\beta-\cos^2\alpha\end{aligned}\right\} \qquad [24]$$

$$\left.\begin{aligned}\cos(\alpha+\beta)\cos(\alpha-\beta)&=\cos^2\alpha-\sin^2\beta\\&=\cos^2\beta-\sin^2\alpha\end{aligned}\right\} \qquad [25]$$

§ **50.** *Given the functions of any angle, to find the functions of twice that angle.*

Substituting in [18 *a*] and [19 *a*] α for β, we have,—

$$\left.\begin{aligned}\sin(\alpha+\alpha)=\sin 2\alpha&=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha\\&=2\sin\alpha\cos\alpha\end{aligned}\right\} \qquad [26]$$

$$\left.\begin{aligned}\cos(\alpha+\alpha)=\cos 2\alpha&=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha\\&=\cos^2\alpha-\sin^2\alpha \qquad \text{(I.)}\\&=2\cos^2\alpha-1 \qquad \text{(II.)}\\&=1-2\sin^2\alpha \qquad \text{(III.)}\end{aligned}\right\} \qquad [27]$$

Substituting in [20] and [21] α for $\pm\beta$, we have,—

$$\tan(\alpha+\alpha)=\tan 2\alpha=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}=\frac{2\tan\alpha}{1-\tan^2\alpha} \qquad [28]$$

$$\operatorname{ctn}(\alpha+\alpha)=\operatorname{ctn}2\alpha=\frac{\operatorname{ctn}\alpha\operatorname{ctn}\alpha-1}{\operatorname{ctn}\alpha+\operatorname{ctn}\alpha}=\frac{\operatorname{ctn}^2\alpha-1}{2\operatorname{ctn}\alpha} \qquad [29]$$

§ **51.** *Given the functions of any angle to find the functions of half that angle.*

All the preceding formulas are perfectly general, and will apply as well to one angle as to another. We can therefore substitute $\frac{1}{2}\alpha$ throughout in place of α. Making this substitution, —

in [27 III.] we have, —

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \quad \therefore \sin\frac{\alpha}{2} = \sqrt{\tfrac{1}{2}(1-\cos\alpha)} \qquad [30]$$

in [27 II.] we have, —

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \quad \therefore \cos\frac{\alpha}{2} = \sqrt{\tfrac{1}{2}(1+\cos\alpha)} \qquad [31]$$

By [2], [30], and [31], —

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad [32]$$

and by [3], —

$$\operatorname{ctn}\frac{\alpha}{2} = \frac{1}{\tan\frac{\alpha}{2}} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} \qquad [33]$$

§ **52.** By [29], —

$$\operatorname{ctn} 2\alpha = \frac{\operatorname{ctn}^2\alpha - 1}{2\operatorname{ctn}\alpha} = \frac{1}{2}\,\frac{\operatorname{ctn}^2\alpha - 1}{\operatorname{ctn}\alpha} = \tfrac{1}{2}(\operatorname{ctn}\alpha - \tan\alpha) \quad [34]$$

By [1] and [26], —

$$\csc 2\alpha = \frac{1}{\sin 2\alpha} = \frac{1}{2\sin\alpha\cos\alpha} = \tfrac{1}{2}\sec\alpha\csc\alpha$$

$$= \text{by [4]}\ \frac{\sin^2\alpha + \cos^2\alpha}{2\sin\alpha\cos\alpha} = \tfrac{1}{2}(\tan\alpha + \operatorname{ctn}\alpha) \quad [35]$$

By subtracting [34] from [35] we get, —

$$\tan\alpha = \csc 2\alpha - \operatorname{ctn} 2\alpha.$$

Now, substituting $\frac{1}{2}\alpha$ for α, we get, —

$$\tan\tfrac{1}{2}\alpha = \csc\alpha - \operatorname{ctn}\alpha. \qquad [36]$$

By adding [34] to [35], and substituting as above, we get, —

$$\operatorname{ctn}\tfrac{1}{2}\alpha = \csc\alpha + \operatorname{ctn}\alpha = \frac{1}{\csc\alpha - \operatorname{ctn}\alpha}. \qquad [37]$$

Substituting $45° + \frac{1}{2}\alpha$, and $45° - \frac{1}{2}\alpha$, for $\frac{1}{2}\alpha$ in [36], and we get, —

$$\begin{aligned}\tan(45° + \tfrac{1}{2}\alpha) &= \csc(90° + \alpha) - \operatorname{ctn}(90° + \alpha)\\ &= \text{by [13] } \sec\alpha + \tan\alpha. \qquad [38]\end{aligned}$$

and

$$\tan(45° - \tfrac{1}{2}\alpha) = \csc(90° - \alpha) - \operatorname{ctn}(90° - \alpha)$$

$$= \operatorname{ctn}(45° + \tfrac{1}{2}\alpha) = \sec\alpha - \tan\alpha = \frac{1}{\sec\alpha + \tan\alpha} \qquad [39]$$

§ **53.** Some formulas will now be obtained, the usefulness of which will be seen hereafter.

By [18 *a*] and [18 *b*], —

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta \qquad [40]$$

and $$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta \qquad [41]$$

By [19 *a*] and [19 *b*], —

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta \qquad [42]$$

and $$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta \qquad [43]$$

Put $\alpha+\beta=x \therefore 2\alpha=x+y \therefore \alpha=\frac{1}{2}(x+y)$

$\alpha-\beta=y \quad 2\beta=x-y \therefore \beta=\frac{1}{2}(x-y)$

Substituting these values of α and β in [40], [41], [42], and [43], we get, —

$$\sin x+\sin y=2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$
$$\sin x-\sin y=2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$
$$\cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$
$$\cos y-\cos x=2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

Since these last formulas are perfectly general, we can write any other letters, and therefore α and β, in place of x and y, as follows: —

$$\sin\alpha+\sin\beta=2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \qquad [44]$$
$$\sin\alpha-\sin\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta) \qquad [45]$$
$$\cos\alpha+\cos\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \qquad [46]$$
$$\cos\beta-\cos\alpha=2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta) \qquad [47]$$

By [44] and [45] we have, —

$$\left.\begin{aligned}\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}&=\frac{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}\\ &=\tan\tfrac{1}{2}(\alpha+\beta)\operatorname{ctn}\tfrac{1}{2}(\alpha-\beta)=\frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}\end{aligned}\right\} \qquad [48]$$

In like manner we get, —

$$\frac{\cos\alpha-\cos\beta}{\cos\alpha+\cos\beta}=-\tan\tfrac{1}{2}(\alpha+\beta)\tan\tfrac{1}{2}(\alpha-\beta) \qquad [49]$$

$$\frac{\sin\alpha\pm\sin\beta}{\cos\alpha+\cos\beta}=\tan\tfrac{1}{2}(\alpha\pm\beta) \qquad [50]$$

$$\frac{\sin\alpha\mp\sin\beta}{\cos\beta-\cos\alpha}=\operatorname{ctn}\tfrac{1}{2}(\alpha\pm\beta) \qquad [51]$$

EXAMPLES.

1. Find the sines and cosines of the following angles by the aid of formulas 18, 19, and 12: 0°, 90°, 180°, 270°, 360°, $(90° \pm A)$ $(180° \pm A)$ $(270° \pm A)$, $(360° \pm A)$, and $-A$. (To get sin 0°, put $\beta = \alpha$ in [18, *b*].)

2. From the functions of 30° find those of 15°.

3. Prove $\cos(60° + A) + \cos(60° - A) = \cos A$.

4. Prove $\sin(30° + A) + \sin(30° - A) = \cos A$.

5. Prove $\sin A = \dfrac{2 \tan \frac{1}{2} A}{1 + \tan^2 \frac{1}{2} A}$.

6. Find $\sin 3A$ in terms of $\sin A$.

7. Find $\cos 3A$ in terms of $\cos A$.

8. From the functions of 90° find those of 45°.

9. Given $\sin A = \frac{3}{5}$; find $\sin \frac{1}{2} A$; also $\cos \frac{1}{2} A$.

10. Given $2 \sin A = \sin 2A$; find A.

11. Given $\tan 2A = 3 \tan A$; find A.

12. Prove $\tan 50° + \operatorname{ctn} 50° = 2 \sec 10°$.

13. Given $\sin 280° = m$; find $\sin 160°$.

14. Given $\sin \theta + \cos \theta = \dfrac{1}{\sqrt{2}}$; find θ.

15. Given $\tan(45° - \theta) + \operatorname{ctn}(45° - \theta) = 4$; find θ.

16. Prove $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \tan 2A + \sec 2A$.

17. Prove

$$2 \sin^2 A \sin^2 B + 2 \cos^2 A \cos^2 B = 1 + \cos 2A \cos 2B.$$

18. Prove

$$\tan(45° + A) - \tan(45° - A) = 2 \tan 2A.$$

19. Prove $\dfrac{\sin B}{\sin A} = \dfrac{\sin(2A + B)}{\sin A} - 2 \cos(A + B)$.

20. Prove $\dfrac{1 - \tan^2(45° - A)}{1 + \tan^2(45° - A)} = \sin 2A$.

21. Prove $\frac{4 \tan A\, (1 - \tan^2 A)}{(1 + \tan^2 A)^2} = \sin 4\, A.$

22. Prove

$$\sin A\, (1 + \tan A) + \cos A\, (1 + \text{ctn}\, A) = \csc A + \sec A.$$

23. Prove

$$\cos A + \cos\, (120° - A) + \cos\, (120° + A) = 0.$$

24. Prove

$$4 \sin A \sin\, (60° - A) \sin\, (60° + A) = \sin 3\, A.$$

25. Given $\sin 3\, \theta + \sin 2\, \theta + \sin \theta = 0$; find θ.

26. Given $\sin\, (x + a) = \cos\, (x - a)$; find x.

27. Prove $2 \sec 2\, A = \sec\, (45° + A) \sec\, (45° - A)$.

28. Given $\tan \theta = \frac{b}{a}$; find the value of $a \cos 2\, \theta + b \sin 2\, \theta$.

In the next five formulas k denotes any integer. These formulas can be readily proved by making the proper substitutions in formulas 40, 41, 42, 43, and 20.

29. Prove

$$\sin k\, \alpha = 2 \sin\, (k - 1)\, \alpha \cos \alpha - \sin\, (k - 2)\, \alpha.$$

30. Prove

$$\sin k\, \alpha = 2 \cos\, (k - 1)\, \alpha \sin \alpha + \sin\, (k - 2)\, \alpha.$$

31. Prove

$$\cos k\, \alpha = 2 \cos\, (k - 1)\, \alpha \cos \alpha - \cos\, (k - 2)\, \alpha.$$

32. Prove

$$\cos k\, \alpha = - 2 \sin\, (k - 1)\, \alpha \sin \alpha + \cos\, (k - 2)\, \alpha.$$

33. Prove $\tan k\, \alpha = \frac{\tan\, (k - 1)\, \alpha + \tan \alpha}{1 - \tan\, (k - 1)\, \alpha \tan \alpha}.$

34. Prove by the aid of the formula of Ex. 31 $\cos 3\, A = 4 \cos^3 A - 3 \cos A$.

35. Prove that when $\frac{1}{2}$ A is between 0° and 90°, $\sqrt{1+\sin A}=1+2\sin\frac{1}{4}A\sqrt{1-\sin\frac{1}{2}A}$.

36. Prove $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$.

37. Prove $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=45°$.

38. Prove $\sin^{-1}\frac{1}{\sqrt{5}}+\text{ctn}^{-1}3=45°$.

39. Given $\theta=\cos^{-1}\sqrt{\frac{2}{3}}$, and $\varphi=\cos^{-1}\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}$; prove $\varphi+\theta=60°$.

40. Obtain $\sin\frac{A}{2}$ in terms of sin A when $\frac{A}{2}$ lies between $-45°$ and $-135°$.

41. Find cos 105°.

42. Find tan 165°.

43. Determine the limits between which A must lie in order that

$$2\sin A=+\sqrt{1+\sin 2A}-\sqrt{1-\sin 2A}.$$

44. If $\tan\frac{\theta}{2}=\left(\frac{1+c}{1-c}\right)^{\frac{1}{2}}\tan\frac{\varphi}{2}$, show that $\cos\theta=\frac{\cos\varphi-c}{1-c\cos\varphi}$.

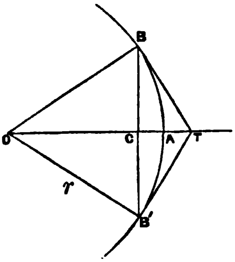
CHAPTER IV.

TRIGONOMETRIC TABLES.

§ 54. *Functions of small angles.*

Let AOB be any positive acute angle, and let α be its circular measure. Let AOB$' = -\alpha$, and therefore B$'$OB $= 2\alpha$; and let O be the centre of a circle of which the radius is $r =$ OA $=$ OB $=$ OB$'$. Let the chord BB$'$ cut OA at C; and let the tangents at B and B$'$ meet at T, which must lie in OA produced. Then,—

FIG. 39.



$$\alpha = \frac{\text{arc AB}}{\text{OA}} = \frac{\text{arc BB}'}{2r}, \qquad \sin\alpha = \frac{\text{CB}}{\text{OA}} = \frac{\text{chord BB}'}{2r},$$

$$\tan\alpha = \frac{\text{TB}}{\text{OB}} = \frac{\text{B}'\text{T}}{\text{OB}'} = \frac{\text{B}'\text{T} + \text{TB}}{2r}.$$

Now, we know by geometry, that

$$\text{chord BB}' < \text{arc BB}' < \text{B}'\text{T} + \text{TB}.$$

Hence, dividing through by $2r$, and substituting the above values, we have,—

$$\sin\alpha < \alpha < \tan\alpha; \qquad [52]$$

or, *the circular measure of an acute angle is greater than the sine of the angle, and less than the tangent of the angle.*

The last written inequality may assume the form, —

$$\frac{1}{\sin\alpha} > \frac{1}{\alpha} > \frac{1}{\tan\alpha}.$$

Now, multiplying through by sin α, we have —

$$1 > \frac{\sin\alpha}{\alpha} > \cos\alpha; \qquad [53]$$

and, multiplying through by sec α, —

$$\sec\alpha > \frac{\tan\alpha}{\alpha} > 1. \qquad [54]$$

But, if $\alpha = 0$, $\cos\alpha = \sec\alpha = 1$; and the differences $1 - \cos\alpha$ (*v.* [53]) and $\sec\alpha - 1$ (*v.* [54]) can be made less than any assigned quantity by taking α sufficiently small. Hence, —

$$\frac{\sin\alpha}{\alpha} = 1 - \varepsilon, \quad \frac{\tan\alpha}{\alpha} = 1 + \varepsilon', \qquad [55]$$

where ε and ε' are positive quantities which decrease indefinitely when α decreases. We have then —

$$\sin\alpha = (1 - \varepsilon)\,\alpha, \quad \tan\alpha = (1 + \varepsilon')\,\alpha; \qquad [56]$$

or, if α is very small, —

$$\sin\alpha = \tan\alpha = \alpha, \text{ approximately}. \qquad [57]$$

For example, —

$$\sin 1' = \tan 1' = \frac{\pi}{10800} = 0.0002908882\ldots.$$

§ **55.** We have, by [30], —

$$1 - \cos\alpha = 2\sin^2\tfrac{1}{2}\alpha.$$

Now, by [55], —

$$\sin\tfrac{1}{2}\alpha = (1 - \varepsilon'') \times \tfrac{1}{2}\alpha = \tfrac{1}{2}(1 - \varepsilon'')\,\alpha;$$

$$\therefore\ 1 - \cos\alpha = \tfrac{1}{2}(1 - \varepsilon'')^2\alpha^2,$$

$$\text{or}\quad \frac{1 - \cos\alpha}{\alpha^2} = \tfrac{1}{2}(1 - \varepsilon'')^2 < \tfrac{1}{2}, \qquad [58]$$

$$\text{or}\quad \cos\alpha = 1 - \tfrac{1}{2}(1 - \varepsilon'')^2\alpha^2 > 1 - \tfrac{1}{2}\alpha^2. \qquad [59]$$

If α is very small, since ε'' decreases indefinitely with α, —

$$\cos\alpha = 1 - \tfrac{1}{2}\alpha^2, \text{ approximately.} \qquad [60]$$

For example, —

$$\cos 1' = 1 - \frac{0.0000000846\ldots.}{2} = 0.999999958\ldots.$$

§ **56.** The formulas proved in the last two sections enable us to compute approximate values of the sine, cosine, and tangent of a very small angle; and the number of decimal places to which these values are correct depends on the smallness of the angle. Let us next seek the limits of error of these formulas, so that we may know in what cases we have a right to use them. Such limits may be found as follows: —

Since $\sin\alpha = 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha$, by [26], and since $\sin\tfrac{1}{2}\alpha > \tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha$, by [53], —

$$\therefore\ \sin\alpha > \alpha\cos^2\tfrac{1}{2}\alpha.$$

But since, by [52], $\sin\tfrac{1}{2}\alpha < \tfrac{1}{2}\alpha$,

$$\therefore\ \cos^2\tfrac{1}{2}\alpha = 1 - \sin^2\tfrac{1}{2}\alpha > 1 - \tfrac{1}{4}\alpha^2;$$

$$\therefore\ \sin\alpha > \alpha(1 - \tfrac{1}{4}\alpha^2) = \alpha - \tfrac{1}{4}\alpha^3. \qquad [61]$$

Again, by [30], $\cos\alpha = 1 - 2\sin^2 \frac{1}{2}\alpha$; and, by the formula just proved, —

$$\sin\tfrac{1}{2}\alpha > \tfrac{1}{2}\alpha - \tfrac{1}{32}\alpha^3;$$

$$\therefore \cos\alpha < 1 - 2\left(\tfrac{1}{2}\alpha - \tfrac{1}{32}\alpha^3\right)^2 < 1 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{16}\alpha^4. \qquad [62]$$

For example,

$$\text{if } \alpha = \text{circ. meas. } 1' = \frac{\pi}{10800} = 0.000290888\ldots.,$$

$$\tfrac{1}{4}\alpha^3 = 0.000000000006\ldots.,$$

$$\tfrac{1}{16}\alpha^4 = 0.0000000000000004\ldots.;$$

so that the above methods are capable of giving values of sin 1′ accurate to *eleven* places of decimals, and of cos 1′ accurate to *fifteen* places of decimals.

By methods of the higher mathematics, we can establish still smaller limits of error; proving that

$$\left.\begin{aligned}\sin\alpha &> \alpha - \tfrac{1}{6}\alpha^3,\\ \cos\alpha &< 1 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{24}\alpha^4.\end{aligned}\right\} \qquad [63]$$

Now we easily compute

$\frac{1}{6}\alpha^3 < 0.00001$, if $\alpha < 0.03915$ = circ. meas. 2° 14′.6,
$\frac{1}{24}\alpha^4 < 0.00001$, if $\alpha < 0.12447$ = circ. meas. 7° 7′.9.

Hence, up to these limits respectively, we may use the formulas $\sin\alpha = \alpha$, $\cos\alpha = 1 - \frac{1}{2}\alpha^2$, with an error less than a unit in the fifth place of decimals.

§ 57. *To show how a table of trigonometric functions can be computed.*

It is necessary to find the functions of angles in the first quadrant only, as the functions of angles in the other quadrants are easily deduced from these. Since the sine of an angle is the cosine of its complement, a complete table of sines will also be a complete table of cosines; and from the sine and cosine of an angle the other functions are readily computed. We need only show, then, how to construct a complete table of sines for the first quadrant.

We may use the formula $\sin\alpha = \alpha$ for small values of α, up to a certain limit depending on the number of decimal places used in the table, and determined as in the last section. We may then proceed by the following method. By [40], —

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta.$$

Let α be any angle, and let β be so small that we can use the formula [60], —

$$\cos\beta = 1-\tfrac{1}{2}\beta^2$$

with an error far below the greatest error allowed in the table; for example, let β = circ. meas. 1′. Then the above formula gives, —

$$\sin(\alpha+\beta)=2\sin\alpha-\sin(\alpha-\beta)-\beta^2\sin\alpha,$$

or

$$\sin(\alpha+\beta)-\sin\alpha=\sin\alpha-\sin(\alpha-\beta)-\beta^2\sin\alpha. \quad [64]$$

If, then, we know $\sin\alpha$ and $\sin(\alpha-\beta)$, we can compute $\sin(\alpha+\beta)$; then from $\sin(\alpha+\beta)$ and $\sin\alpha$, we can get $\sin(\alpha+2\beta)$; and so on, using the successive formulas, —

$$\sin(\alpha+\beta)-\sin\alpha=\sin\alpha-\sin(\alpha-\beta)-\beta^2\sin\alpha,$$

$$\sin(\alpha+2\beta)-\sin(\alpha+\beta)=\sin(\alpha+\beta)-\sin\alpha-\beta^2\sin(\alpha+\beta),$$

$$\sin(\alpha+3\beta)-\sin(\alpha+2\beta)=\sin(\alpha+2\beta)-\sin(\alpha+\beta)-\beta^2\sin(\alpha+2\beta),\ \text{\&c.}$$

If $\beta =$ circ. meas. $1' = 0.00029\ldots$, $\beta^2 = 0.0000000846\ldots$, and, as this small number is a constant multiplier in the last term, computation by the above formulas is quite an easy process.

By the successive application of these formulas, we may find the sines of all angles of the quadrant at any small interval β; for example, 1′.

§ **58.** Given $\sin 20° = 0.34202$, $\sin 20° 30' = 0.35021$; let the student compute the following values by the above formulas, assuming $\beta =$ circ. meas. 30′: —

$$\sin 21° = 0.3584,$$

$$\sin 21° 30' = 0.3665,$$

$$\sin 22° = 0.3746,$$

$$\sin 22° 30' = 0.3827,$$

$$\sin 23° = 0.3907.$$

§ **59.** When the values of the sine have been computed up to 60°, its values for the rest of the quadrant can be easily found by the formula, —

$$\sin (60° + \alpha) - \sin (60° - \alpha) = 2 \cos 60° \sin \alpha = \sin \alpha,$$

or

$$\sin (60° + \alpha) = \sin (60° - \alpha) + \sin \alpha.$$

Again, we have found in §§ 38, 39, the values of the trigonometric functions for 30°, 45°, and 60°; and from these we can compute those of 22° 30′, 15°, 11° 15′, 7° 30′, 5° 37′.5, 3° 45′, &c. The results thus obtained may be used as tests of the computation made by the method of § 57.

§ **60.** The trigonometric functions are called *natural* trigonometric functions, to distinguish them from their logarithms, which are called *logarithmic* trigonometric functions.

Tables of the natural and logarithmic trigonometric functions have been constructed; and a thorough understanding of the use of these tables is necessary for performing trigonometric computations.

Since there is some variation in the arrangement of different tables, the student can better learn their use by referring to the explanations which accompany the tables, than by any explanations which could be given here.

Peirce's four-place tables will be used in the solution of problems in the following chapters.

§ **61.** *Interpolation.* A trigonometric table cannot contain the functions of *all* angles of the quadrant, but only of angles separated by a certain interval; for example, an interval of 1′ or of 10′. The functions of intermediate angles can be found by the same principle of *proportional parts* which is applied to the table of logarithms of numbers. We can prove the validity of this principle for the trigonometric functions as follows:—

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

Now, if β is very small, we can by [60] and [57], take $\cos\beta=1$, $\sin\beta=\beta$, and thus we have

$$\sin(\alpha+\beta)=\sin\alpha+\beta\cos\alpha,\text{ approximately,}$$

$$\cos(\alpha+\beta)=\cos\alpha-\beta\sin\alpha,\text{ approximately.}$$

Hence, if β and β' are two very small angles, —

$$\frac{\sin(\alpha+\beta)-\sin\alpha}{\sin(\alpha+\beta')-\sin\alpha}=\frac{\beta\cos\alpha}{\beta'\cos\alpha}=\frac{\beta}{\beta'}, \text{ approximately.}$$

$$\frac{\cos\alpha-\cos(\alpha+\beta)}{\cos\alpha-\cos(\alpha+\beta')}=\frac{\beta\sin\alpha}{\beta'\sin\alpha}=\frac{\beta}{\beta'}, \text{ approximately.}$$

So that, for very small variations of an angle, the variations of the sine and cosine are approximately proportional to the variations of the angle. Now, it is a known principle of logarithms, that, for very small variations of a number, the variations of the logarithm are approximately proportional to those of the number (*Peirce's Elements of Logarithms*, § 15). Hence, for very small variations of an angle, the variations of the log sin and log cos are approximately proportional to those of the angle.

The same principle can be extended, by similar proofs, to the other natural and logarithmic trigonometric functions. The *limits* within which it may be properly employed can also be investigated by means of the formulas of §§ 55 and 56.

§ **62.** It must be remembered that tables of logarithms and of trigonometric functions only give the values of the functions to a certain number of decimal places. For example: log sin 16° is given by our four-place tables as 9.4403; but this only informs us that the true value lies somewhere between 9.44025 and 9.44035. Thus every logarithm (and every natural function as well) taken from a four-place table is liable to an error not exceeding ± 0.00005; and, if the function is found by interpolation, its total error may have any value short of 0.0001. If, then, a logarithm is obtained by adding or subtracting several logarithms, the error of the resulting logarithm *may* be several times 0.00005.

"When a change of 0.0001 in any logarithmic trigonometric function produces a change of more than 1′ in the corresponding angle, the *four-place logarithm does not determine the angle to minutes*, since the uncertainty of ± 0.00005 in the logarithm will leave an uncertainty of ± 0′.5 in the angle. This will never be the case with the log tan or log ctn.

"It is to be observed, that, with four-place logarithms, the angle can be determined to minutes from its *log tan* or *log ctn in all parts of the quadrant; from its log sin or log csc* in the *first half* of the quadrant; and from its *log cos* or *log sec* in the *last half.* This observation will often direct the computer in the choice of his method, where several lie open to him."— *Peirce's Elements of Logarithms.*

CHAPTER V.

SOLUTION OF RIGHT TRIANGLES.

§ **63.** To *solve* a triangle is to obtain from its given parts the values of its unknown parts. In every triangle there are six parts; namely, three angles and three sides.

We have learned from geometry, that, in general, it is possible to *construct* a triangle when any three of its parts are given, provided one of them is a side. We shall now show, that, in general, it is also possible to solve a triangle when we have the above data.

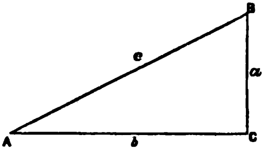
In order to solve a triangle, we must obtain formulas by which each unknown part can be obtained from the known parts.

As we proceed, we shall find that there are often several formulas by which any particular part may be found. A formula should be selected which is adapted to *logarithmic computation*, and which will give accurate results; and it is also well, when possible, to use formulas by which each unknown part can be obtained directly from those given at the outset.

§ **64.** "After a table of logarithms has once been constructed, the labor of certain arithmetic operations can be materially diminished, while at the same time the chance of committing errors is lessened. The sole arithmetical operations which can be performed by logarithms are those of multiplication, division, involution, and evolution. Before, therefore, the value of an expression can be calculated by means of logarithms, the expression must be put into such a form that no other arithmetic operations than these have to be performed. Such an arrangement of an expression is called the *adaptation* of it to *logarithmic* computation." — *Snowball.*

Thus, when in a right triangle (Fig. 40) a and c are given, and b is required, logarithms cannot be *directly* applied to determine the value of b from the equation $b^2 = c^2 - a^2$ or $b = \sqrt{c^2 - a^2}$ (§ 21, e) ; but if the equation is put into the following form, $b = \sqrt{(c + a)\ (c - a)}$, logarithms can be directly applied to determine the value of b.

FIG. 40.



When a and b are given, and c is required, the formula $c^2 = a^2 + b^2$ not being adapted to logarithmic computation, will not, in general, be used to determine c, but the problem will be solved in a manner to be explained hereafter.

Since, as has been stated above, it is generally advantageous to make use of logarithms in problems involving arithmetic operations, care will be taken in this, and in the following chapter, to deduce and make use of formulas adapted to logarithmic computation.

§ **65.** When we know that one of the angles of a triangle is 90°, i.e., that the triangle is a right triangle, we can solve it by simpler methods than when all of the angles are oblique; we shall therefore consider right triangles separately, although they can be solved perfectly well by the general formulas which will be obtained hereafter.

When, in addition to the right angle, a side and one other part of a right triangle are given, it can be solved by formulas with which the student is already familiar, but which, for the sake of convenience, will be collected below.

$$A+B=90^\circ \quad \text{(I.)}$$

$$\left.\begin{array}{l}\sin A=\frac{a}{c}\\ \csc A=\frac{c}{a}\end{array}\right\}\text{(II.)} \quad \left.\begin{array}{l}\cos A=\frac{b}{c}\\ \sec A=\frac{c}{b}\end{array}\right\}\text{(III.)} \quad \left.\begin{array}{l}\tan A=\frac{a}{b}\\ \operatorname{ctn} A=\frac{b}{a}\end{array}\right\}\text{(IV.)}$$

$$c^2=a^2+b^2 \quad \left\{\begin{array}{l}c^2-a^2=b^2=(c+a)(c-a)\\ c^2-b^2=a^2=(c+b)(c-b)\end{array}\right.$$

$$\therefore \begin{array}{l}b=\sqrt{(c+a)\ (c-a)}\\ a=\sqrt{(c+b)\ (c-b)}\end{array} \qquad [65]$$

§ **66.** It will be worth while to begin by considering what cases can arise in the solution of right triangles. One side must be given, and may be either a leg or the hypothenuse. If a leg is given, there may be joined with it either the other leg, the hypothenuse, or an acute angle. If the hypothenuse is given, there may be joined with it either a leg (a case already considered) or an acute angle.

The following, then, are all the cases which can arise: —

Case I. Given the two legs.
Case II. Given a leg and the hypothenuse.
Case III. Given a leg and an acute angle.
Case IV. Given the hypothenuse and an acute angle.

In each of these cases, any one of the required parts may be obtained from the given parts by using that one of the formulas in § 65 which involves the parts in question.

The general rule given above should be rigorously followed, except in those cases where it will give formulas which are not adapted to logarithmic computation, or which will not give accurate results.

§ **67**. Case I. When we wish to find the hypothenuse from the two legs, the formula which we should choose in accordance with the rule, — i.e., $c^2 = a^2 + b^2$, — is not adapted to logarithmic computation. Here, then, we we shall have to deviate from the rule; and, in finding the hypothenuse, we will proceed as follows: we will first find the angles by the formula, $\tan A = \operatorname{ctn} B = \frac{a}{b}$; and having found the angles we will get c from the formula, $\csc A = \frac{c}{a}$, or $\csc B = \frac{c}{b}$.

If either angle is very small (say less than 10°), it is best to employ the csc of the *larger* angle; since a small angle must be given with great precision in order to determine accurately its sine and cosecant.

Example I. Given $a = 643.6$, $b = 59.87$.

$$10 + \log\tan A = \log a + \operatorname{colog} b = 10 + \log\operatorname{ctn} B,$$

$$\log c \quad = \log a + \log\csc A,$$

$$\begin{array}{lll|lll}
\log & a = & 2.8086 & \log & a = & 2.8086 \\
\operatorname{colog} & b = & \underline{8.2228} & \log\csc A & = & \underline{0.0019} \\
\log\tan A & = & 1.0314 = \log\operatorname{ctn} B, & \log & c = & 2.8105, \\
\therefore & A = & 84^\circ\,41',\ B = 5^\circ\,19' & & c = & 646.4
\end{array}$$

Example II. Given $a = 0.0227$, $b = 0.0840$.

Answers. $A = 15^\circ\,7'$; $B = 74^\circ\,53'$; $c = 0.0870$.

§ **68.** CASE II. Given a *leg and the hypothenuse* (c), we will denote the given leg by b.

Proceeding in accordance with the rule, we shall get the formulas

$$\sin B = \cos A = \frac{b}{c}, \text{ and } a = \sqrt{(c+b)(c-b)},$$

by which the following examples can be solved: —

Example I. Given $c = 7.458$, $b = 0.6473$.

Answers. $a = 7.430$, $A = 85° 1'$, $B = 4° 59'$.

Example II. Given $c = 672.3$, $b = 548.9$.

Answers. $a = 388.2$, $A = 35° 17'$, $B = 54° 43'$.

The above method is the best when b differs considerably from c; but, when b is nearly equal to c, sin B and cos A will be nearly equal to unity; B will therefore be nearly equal to 90°, and A will be nearly equal to 0°. When this is the case neither A nor B can be obtained with great accuracy from the formula $\sin B = \cos A = \frac{b}{c}$ (*v.* § 62 end).

We will now deduce a new formula by which, when b is nearly equal to c, we can find A (and therefore its complement B) more accurately than by the above method.

From [30], —

$$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A = 1 - \frac{b}{c} = \frac{c-b}{c},$$

$$\therefore \sin \tfrac{1}{2} A = \sqrt{\frac{c-b}{2c}} \qquad [66]$$

Example III. Given $b = 7.4169$, $c = 7.4451$.

From [66], —

$$\log \sin \tfrac{1}{2} A = \tfrac{1}{2} [\log (c-b) + \operatorname{colog} 2c].$$

The formula for finding a is, in accordance with the rule, $a=\sqrt{(c+b)\ (c-b)}$.

$$\therefore\ \log a=\tfrac{1}{2}\left[\log (c+b)+\log (c-b)\right]$$

$$\begin{array}{rrr}
2c=14.8902 & c+b=14.8620 & \log(c+b)=1.1722 \\
\log 2c=1.1730 & c-b=\ 0.0282 & \log(c-b)=\bar{2}.4502 \\
\cline{1-1}\cline{3-3}
\log(c-b)=\bar{2}.4502 & & 2\,|\,\bar{1}.6224 \\
\text{colog}\ 2c=8.8270 & & \log a=\bar{1}.8112 \\
\cline{1-1}
2\,|\,17.2772 & & a=0.6474 \\
\cline{1-1}
\log\sin\tfrac{1}{2}\mathrm{A}=8.6386 & &
\end{array}$$

$\tfrac{1}{2}\mathrm{A}=2^\circ\ 30'$, $\mathrm{A}=5^\circ\ 00'$, $90^\circ-\mathrm{A}=\mathrm{B}=85^\circ\ 00'$.

Example IV. Given $c=84.32$, $b=84.28$.

Answers. $a=2.597$, $\mathrm{A}=1^\circ\ 46'$, $\mathrm{B}=88^\circ\ 14'$.

§ 69. CASE III. *Given a leg and an acute angle.*

In this case and the following case, the unknown angle is the complement of the given angle; while the unknown sides may be found by employing those functions of the given angle of which the numerators are the unknown sides, and the denominator the given side.

Example I. Given $b=0.084$, $\mathrm{A}=15^\circ\ 9'$.

Answers. $c=0.0870$, $a=0.0227$, $\mathrm{B}=74^\circ\ 51'$.

Example II. Given $a=64.82$, $\mathrm{A}=10^\circ\ 3'$.

Answers. $c=371.4$, $b=365.8$, $\mathrm{B}=79^\circ\ 57'$.

§ 70. CASE IV. *Given the hypothenuse and an acute angle.*

Example I. Given $c=426.7$, $\mathrm{A}=34^\circ\ 15'$.

Answers. $a=240.1$, $b=352.7$, $\mathrm{B}=55^\circ\ 45'$.

Example II. Given $c=371.4$, $\mathrm{A}=10^\circ\ 3'$.

Answers. $\mathrm{B}=79^\circ\ 57'$, $b=365.8$, $a=64.82$.

EXAMPLES.

1. Given $c = 0.8423$ $A = 42^\circ\, 28'$ $C = 90^\circ$ solve
2. " $c = 0.4497$ $B = 51^\circ\, 25'$ $C = 90^\circ$ "
3. " $a = 984.2$ $A = 22^\circ\, 59'$ $C = 90^\circ$ "
4. " $b = 6.732$ $A = 79^\circ\, 13'$ $C = 90^\circ$ "
5. " $a = 31.$ $B = 86^\circ\, 18'$ $C = 90^\circ$ "
6. " $b = 984.2$ $B = 22^\circ\, 59'$ $C = 90^\circ$ "
7. " $c = 73$ $b = 55$ $C = 90^\circ$ "
8. " $c = 9$ $b = 8$ $C = 90^\circ$ "
9. " $c = 1006.$ $a = 1000$ $C = 90^\circ$ "
10. " $c = 86$ $a = 17$ $C = 90^\circ$ "
11. " $a = 0.1$ $b = 1.069$ $C = 90^\circ$ "
12. " $a = 272$ $b = 224$ $C = 90^\circ$ "

13. Given one leg equal to .7, and the angle opposite this leg equal to $\frac{2}{3}$ the angle opposite the other leg: solve.

14. If one leg of a right triangle is $\frac{7}{8}$ the hypothenuse, what are the angles of the triangle? and what is the ratio of the other leg to the hypothenuse?

15. Given one leg (a) and the perpendicular from the right angle on to the hypothenuse: solve.

16. Having measured a distance of 200 feet in a direct horizontal line from the bottom of a steeple, the angle of elevation of its top was found to be 46° 30′: find the height of the steeple. *Ans.* 210.8 ft.

17. A river whose breadth AC is 200 feet runs at the foot of a tower CB which subtends an angle BAC of 25° 10′ at the edge of the bank: find the height of the tower. *Ans.* 93.98 ft.

18. A person on the top of a tower 50 feet high observes the angle of depression of two objects on the horizontal plane, which are in the same straight line with the tower, to be 30° and 45°: find their distances from each other and from the observer.

19. A tower 150 feet high throws a shadow 75 feet long upon the horizontal plane on which it stands: find the sun's altitude. *Ans.* 63° 26′.

20. A tower stands by a river. A person on the opposite bank finds its elevation to be 60°; he recedes 40 yards in a direct line from the tower, and then finds the elevation to be 50°: find the breadth of the river.

21. A rope is fastened to the top of a building 60 feet high. The length of the rope is 109 feet: find the angle at which it is inclined to the horizon. *Ans.* 33° 24′.

22. Standing straight in front of a house, opposite one corner, I find that its length subtends an angle whose tangent is 2, while its height subtends an angle whose tangent is $\frac{3}{8}$; the height of the house is 45 feet: find its length.

23. From a balloon which is directly above one town, is observed the angle of depression of another town, 10° 14′. The towns being 8 miles apart, find the height of the balloon. *Ans.* 1.444 miles.

24. Given an angle and the area: find formulas for the three sides. E. g. $A = 38° 27'$, area 158.4.

25. Prove that in a right triangle, area $= s(s - c)$, where C is the right angle, and $s = \frac{1}{2}(a + b + c)$.

26. Wishing to know the height of an inaccessible hill, I took the angle of elevation of its top to be 60°; I then measured 100 feet away from the hill, and found the angle of elevation to be 45°: what is the height of the hill?

27. Find the angle which a flagstaff 5 yards long, and standing on the top of a tower 200 yards high, subtends at a point in the horizontal plane 100 yards from the base of the tower. *Ans.* 0° 34′.

28. The depth of a dam is 8 feet; the width at the top is 10 feet; the inclination to the horizon of each side is 38° 53′. Find the width of the dam at the bottom.

Ans. 29.84 feet.

29. When the altitude of the sun is 30°, the length of the shadow cast by Bunker Hill Monument is 381.1 feet. Find the height of the monument. *Ans.* 220 feet.

30. The earth's radius is 3,963 miles. Find the radius of the circle of latitude through Cambridge, Mass. (lat. 42° 23′). *Ans.* 2,927 miles.

31. Find the length, in geographical miles,[*] of the arc of the circle of latitude joining Halifax (long. 63° 35′ W.; lat. 44° 40′ N.) with Cape Ferret, near Bordeaux, France (long. 1° 14′ W.; lat. same as that of Halifax).

Ans. 2,661 miles.

32. A regular heptagon is inscribed in a circle the radius of which is 6.6. Find a side and the area of the heptagon. *Ans.* Side, 5.726; area, 119.2.

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[*] A geographical mile is the length of an arc of 1′ measured on the earth's equator.

CHAPTER VI.

FORMULAS RELATING TO OBLIQUE TRIANGLES. — SOLUTION OF OBLIQUE TRIANGLES.

§ **71.** *The sides of any triangle are proportional to the sines of the opposite angles.*

FIG. 41.

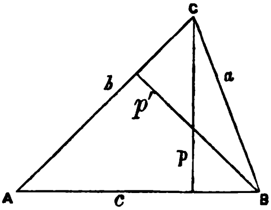
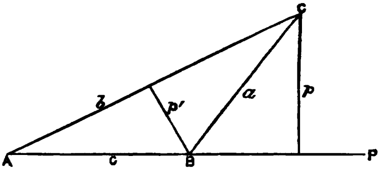


FIG. 42.



From B and C (Figs. 41 and 42) let fall the perpendiculars p' and p on b and c respectively. Then

$$\sin A = \frac{p}{b}, \ \sin B = \frac{p}{a} \ \therefore \ \frac{\sin A}{\sin B} = \frac{p}{b} \times \frac{a}{p} = \frac{a}{b}, \qquad (\alpha)$$

and $$\sin A = \frac{p'}{c}, \ \sin C = \frac{p'}{a} \ \therefore \ \frac{\sin A}{\sin C} = \frac{p'}{c} \times \frac{a}{p'} = \frac{a}{c}. \qquad (\beta)$$

From (α) we get, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, From ($\beta$) we get, $\frac{a}{\sin \text{A}}=\frac{c}{\sin \text{C}}$,

$$\therefore \frac{a}{\sin \text{A}}=\frac{b}{\sin \text{B}}=\frac{c}{\sin \text{C}}. \qquad [67]$$

In applying the above to Fig. 42, we must remember that sin B is equal to the sine of its supplement CBP.

§ **72.** *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

From [67] we get, by the theory of proportions, —

$$\frac{a+b}{a-b}=\frac{\sin \text{A}+\sin \text{B}}{\sin \text{A}-\sin \text{B}}=\text{by [48]}\ \frac{\tan \frac{1}{2}(\text{A}+\text{B})}{\tan \frac{1}{2}(\text{A}-\text{B})}, \quad [68]$$

In like manner

$$\frac{a+c}{a-c}=\frac{\tan \frac{1}{2}(\text{A}+\text{C})}{\tan \frac{1}{2}(\text{A}-\text{C})}, \text{ and } \frac{b+c}{b-c}=\frac{\tan \frac{1}{2}(\text{B}+\text{C})}{\tan \frac{1}{2}(\text{B}-\text{C})},$$

§ **73.** *The square of any side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those sides into the cosine of their included angle.*

FIG. 43.

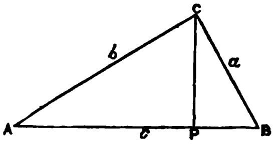
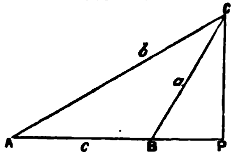


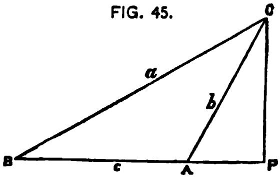
FIG. 44.



Through C in the triangle ABC (Figs. 43, 44, and 45) draw PC perpendicular to c. In each figure, then,

$$a^2=\text{PB}^2+\text{PC}^2,$$

FIG. 45.



but $$\text{PB} = \text{PA} + \text{AB} = -\text{AP} + c \quad (v.\ \S\ 3,\ d.)$$

$$\therefore \text{PB}^2 = \text{AP}^2 - 2\,\text{AP}\,c + c^2,$$

$$\therefore a^2 = \text{PC}^2 + \text{AP}^2 + c^2 - 2\,\text{AP}\,c,$$

but $\cos A = \frac{\text{AP}}{b}$,

$$\therefore \text{AP} = b \cos A, \text{ and } \text{PC}^2 + \text{AP}^2 = b^2.$$

$$\therefore a^2 = b^2 + c^2 - 2\,b\,c \cos A,$$

and in like manner

$$\left.\begin{aligned} b^2 &= c^2 + a^2 - 2\,c\,a \cos B, \\ c^2 &= a^2 + b^2 - 2\,a\,b \cos C. \end{aligned}\right\} \quad [69]$$

Corollary. *Formula for the diagonal of a parallelogram.*

Let w denote an angle of a parallelogram, m and n its sides, d the diagonal which divides w, and w_1 the supplement of w.

By constructing a proper figure it can be readily seen that d, m, and n form a triangle in which the angle included by m and n is w_1.

$\therefore$ by [69], —

$$d^2 = m^2 + n^2 - 2\,m\,n \cos w_1\,;$$

but since w and w_1 are supplements, we have, —

$$\cos w_1 = -\cos w\,;$$

$$\therefore d^2 = m^2 + n^2 + 2\,m\,n \cos w. \qquad [70]$$

§ **74.** *Formula for the side of a triangle in terms of the cosines of the adjacent angles and the other two sides.*

From Figs. 43–45, it is evident that, —

$$\left.\begin{aligned} a &= b \cos C + c \cos B, \\ b &= c \cos A + a \cos C, \\ c &= a \cos B + b \cos A. \end{aligned}\right\} \quad [71]$$

From [71] all the relations between the six parts of a triangle can be deduced by algebraic transformations; but the processes are somewhat longer than those given in the text.

§ **75.** *Formulas for sine and cosine of* $\frac{1}{2}$A, $\frac{1}{2}$B, *and* $\frac{1}{2}$C. From [69] we find

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \text{by [27 III.] } 1-2\sin^2\tfrac{1}{2}A,$$

$$\therefore 2\sin^2\tfrac{1}{2}A = 1-\frac{b^2+c^2-a^2}{2bc} = \frac{a^2-b^2-c^2+2bc}{2bc},$$

$$= \frac{a^2-(b^2-2bc+c^2)}{2bc} = \frac{(a-b+c)(a+b-c)}{2bc},$$

put $2s = a+b+c$, $\therefore a-b+c = 2(s-b)$

and $a+b-c = 2(s-c)$

$$\therefore 2\sin^2\tfrac{1}{2}A = \frac{2(s-b)\,2(s-c)}{2bc}$$

$$\therefore \sin\tfrac{1}{2}A = +^{*}\sqrt{\frac{(s-b)(s-c)}{bc}},$$

and in like manner

$$\left.\begin{aligned} \sin\tfrac{1}{2}B &= +\sqrt{\frac{(s-c)(s-a)}{ca}}, \\ \sin\tfrac{1}{2}C &= +\sqrt{\frac{(s-a)(s-b)}{ab}}, \end{aligned}\right\} \qquad [72]$$

Also, —

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \text{by [27 II.] } 2\cos^2\tfrac{1}{2}A-1,$$

---

* A, B, and C are less than 180°, and $\frac{1}{2}$ A, $\frac{1}{2}$ B, and $\frac{1}{2}$ C are less than 90°: we must then give the + sign to the radicals.

$$\therefore \cos^2 \tfrac{1}{2} A = \frac{b^2 + c^2 - a^2 + 2bc}{4bc} = \frac{(b+c)^2 - a^2}{4bc},$$

$$= \frac{(b+c+a)(b+c-a)}{4bc} = \frac{2s2(s-a)}{4bc} = \frac{s(s-a)}{bc}$$

$$\left.\begin{array}{l} \therefore \cos \tfrac{1}{2} A = +^{*} \sqrt{\dfrac{s(s-a)}{bc}}, \\ \text{and in like manner} \\ \cos \tfrac{1}{2} B = + \sqrt{\dfrac{s(s-b)}{ca}}, \\ \cos \tfrac{1}{2} C = + \sqrt{\dfrac{s(s-c)}{ab}}, \end{array}\right\} \quad [73]$$

Since the sine of an angle and the sine of its supplement are the same (*v.* [8]), whenever all that is given concerning an angle is the value of its sine, the angle may have either of two supplementary values. The ambiguity thus arising in the use of [72] is, however, removed by the consideration, that, since A, B, and C, being angles of a triangle, are each less than 180°, $\frac{1}{2}$ A, $\frac{1}{2}$ B, and $\frac{1}{2}$ C are each less than 90°.

§ **76.** *Formulas for tan* $\frac{1}{2}$ A, *tan* $\frac{1}{2}$ B, *and tan* $\frac{1}{2}$ C, *in terms of* s, a, b, *and* c.

By [2], [72], and [73] —

$$\tan \tfrac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \frac{\sqrt{\dfrac{(s-b)(s-c)}{bc}}}{\sqrt{\dfrac{s(s-a)}{bc}}}$$

---

[*] *v.* note p. 75.

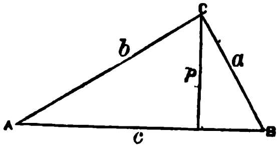
$$\therefore\ \tan \tfrac{1}{2}\mathrm{A} = +^{*}\sqrt{\frac{(s-b)(s-c)}{s\,(s-a)}}$$

and in like manner

$$\left.\begin{aligned}\tan \tfrac{1}{2}\mathrm{B} &= +\sqrt{\frac{(s-c)(s-a)}{s\,(s-b)}}\\ \tan \tfrac{1}{2}\mathrm{C} &= +\sqrt{\frac{(s-a)(s-b)}{s\,(s-c)}}\end{aligned}\right\}\qquad [74]$$

§ **77.** *Formulas for the area of a triangle, in terms of two sides and the included angle; in terms of one side and the adjacent angles; and in terms of s and the three sides.*

FIG. 46.



From geometry we know that the area of a triangle is half the product of its base by its altitude.

$$\therefore\ \text{area of ABC (Fig. 46)} = \tfrac{1}{2}p\,c,$$

but $\quad \sin \mathrm{A} = \dfrac{p}{b}\quad \therefore p = b \sin \mathrm{A}.$

$\therefore$ denoting the area by K we have, —

$$\mathrm{K} = \tfrac{1}{2}\,b\,c \sin \mathrm{A}. \qquad [75]$$

By [67], —

$$b = \frac{a \sin \mathrm{B}}{\sin \mathrm{A}}, \text{ and } c = \frac{a \sin \mathrm{C}}{\sin \mathrm{A}}.$$

Substituting these values in [75], and noting that, since $\mathrm{A} = 180^\circ - (\mathrm{B} + \mathrm{C})$, $\sin \mathrm{A} = \sin(\mathrm{B}+\mathrm{C})$, we get, —

$$\mathrm{K} = \tfrac{1}{2}\,\frac{a^2 \sin \mathrm{B} \sin \mathrm{C} \sin \mathrm{A}}{\sin^2 \mathrm{A}} = \tfrac{1}{2}\,\frac{a^2 \sin \mathrm{B} \sin \mathrm{C}}{\sin(\mathrm{B}+\mathrm{C})}. \qquad [76]$$

---

* *v.* note p. 75.

By [26], —

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A.$$

$\therefore$ substituting in [75] we get, —

$$K = \tfrac{1}{2} b c \, 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A.$$

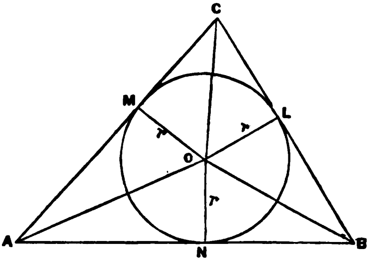
$\therefore$ by [72] and [73], —

$$K = b c \sqrt{\frac{(s-b)(s-c)(s-a)\,s}{bc \cdot bc}}.$$

$$= \sqrt{s\,(s-a)(s-b)(s-c)} \qquad [77]$$

§ **78.** *Formula for the area of a triangle, in terms of s and r (the radius of the inscribed circle).*

FIG. 47



Let O (Fig. 47) be the centre of the inscribed circle; then each of the lines OL, OM, and ON, perpendicular to a, b, and c respectively, will be equal to r.

Now, the triangle

$$ABC = BOC + COA + AOB = \tfrac{1}{2} a r + \tfrac{1}{2} b r + \tfrac{1}{2} c r.$$

$$\therefore \; K = \tfrac{1}{2}(a + b + c)\, r = s r \qquad [78]$$

§ **79.** *Formula for the radius of the inscribed circle, in terms of s, a, b, and c; and in terms of s, A, B, and C.*

By [78] and [77], —

$$r = \frac{K}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad [79]$$

From [79] and [74] we get, —

$$r = s \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C. \qquad [80]$$

§ **80.** *Formulas for* $\tan \frac{1}{2} A$, $\tan \frac{1}{2} B$, *and* $\tan \frac{1}{2} C$, *in terms of* r, s, a, b, *and* c.

Dividing both sides of [79] by $s - a$, we get, —

$$\frac{r}{s-a} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

∴ by [74], —

$$\left.\begin{aligned} \tan \tfrac{1}{2} A &= \frac{r}{s-a} \\ \text{and in like manner} \quad \tan \tfrac{1}{2} B &= \frac{r}{s-b} \\ \tan \tfrac{1}{2} C &= \frac{r}{s-c} \end{aligned}\right\} \qquad [81]$$

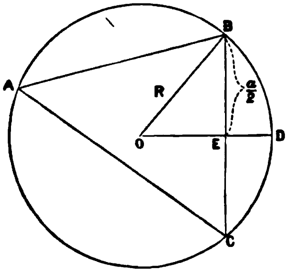
§ **81.** *Formulas for the perpendiculars* p_a, p_b, *and* p_c, *from the angles upon the sides* a, b, *and* c *respectively.*

$$\left.\begin{aligned} p_a = b \sin C = c \sin B &= \text{by } [67] \; \frac{bc}{a} \sin A \\ \therefore \text{ by } [75] \; p_a &= \frac{2K}{a} \\ \text{and in like manner} \quad p_b &= \frac{2K}{b} \\ p_c &= \frac{2K}{c} \end{aligned}\right\} \qquad [82]$$

§ 82. *Formulas for radius of circumscribed circle* (R), *in terms of a side and the opposite angle; and in terms of* K, *a*, *b*, *and c.*

In Fig. 48 draw the radius OD perpendicular to BC; therefore by geometry, —

FIG 48.



$$\text{EB} = \tfrac{1}{2}\,\text{CB} = \tfrac{1}{2}\,a,$$

and $$\text{arc DB} = \tfrac{1}{2}\,\text{arc CB};$$

$$\therefore \text{ the angle DOB} = \text{A}.$$

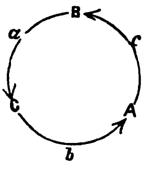
$$\therefore \csc \text{DOB} = \csc \text{A} = \frac{\text{R}}{\frac{1}{2}a}$$

$$\therefore \text{R} = \tfrac{1}{2}\,a \csc \text{A}. \qquad [83]$$

By [75], —

$$\sin \text{A} = \frac{2\,\text{K}}{b\,c} = \frac{1}{\csc \text{A}} \quad \therefore \csc \text{A} = \frac{b\,c}{2\,\text{K}}$$

$$\therefore \text{by [83] R} = \frac{a\,b\,c}{4\,\text{K}} \qquad [84]$$



NOTE. — In each set of formulas numbered 67, 69, 71, 72, 73, 74, and 82, the first is perfectly general: from it, therefore, the other two can be obtained by direct substitution. The work of substitution is made easier by the following rule: Given the first in each set, the others are obtained by advancing the letters; that is, by substituting for each letter of the equation the next letter in the fixed order indicated by the figure in the margin. In this manner, the second of [74], for example, can be obtained from the first, and then

§ **83.** *Solution of oblique triangles.*

"We may remark here, that when an angle of a triangle is determined from its cosine, versed sine, tangent, cotangent, or secant, no uncertainty can exist about the angle, because only one angle exists less than 180° for which any of these functions has an assigned value. But when an angle of a triangle is determined from its sine or cosecant, uncertainty may exist, since there are two angles less than 180° which have a given sine or a given cosecant."—*Todhunter*.

There is, however, no uncertainty in the case of right triangles, because each angle, except the right angle, is acute

According to the methods of solution which we shall adopt, there is only one case of oblique triangles where there can be any uncertainty in the angle: this case will be fully discussed hereafter (*v.* Case II. § 85).

§ **84.** CASE I. *Given two angles and a side,—A, B, and a. find C, b, and c.*

$$A+B+C=180^\circ, \therefore C=180^\circ-(A+B).$$

the third from the second, and the first again from the third. In like manner, the second may be obtained from the third, the first from the second, and the third from the first, by moving the letters back.

By the above rule, we can get from [75], —

$$K=\tfrac{1}{2}\,c\,a\sin B,$$

and $$K=\tfrac{1}{2}\,a\,b\sin C.$$

Also from [83] we can get, —

$$R=\tfrac{1}{2}\,b\csc B,$$

and $$R=\tfrac{1}{2}\,c\csc C.$$

By [67],

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}, \therefore b=\frac{a \sin B}{\sin A}=a \sin B \csc A,$$

and $$c=\frac{a \sin C}{\sin A}=a \sin C \csc A;$$

$$\therefore \log b=\log a+\log \sin B+\log \csc A,$$

and $$\log c=\log a+\log \sin C+\log \csc A.$$

Example I.

Given $A=36° 56'$, $B=72° 6'$, $a=36.74$

$\log \csc A$	$=0.2212$	$\log \csc A$	$=0.2212$
$\log \sin B$	$=9.9784$	$\log \sin C$	$=9.9756$
$\log a$	$=1.5652$	$\log a$	$=1.5652$
$\log b$	$=1.7648$	$\log c$	$=1.7620$
b	$=58.19$	c	$=57.81$

$A+B=109° 2'$ $\qquad 180°-(A+B)=C=70° 58'$

Example II.

Given $a=0.3578$, $B=32° 41'$, $C=47° 54'$.

Answers.

$A=99° 25'$, $b=0.1959$, $c=0.2691$.

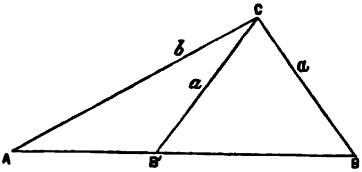
§ **85.** CASE II. *Given two sides and an angle opposite one of them,—a, b, and* A*: find* c, B, *and* C.

It has already been stated (§ 63), that, *in general*, a triangle can be solved when any three of its parts are given, provided one of the given parts is a side. It will now be shown, that, in the present case, the given parts can be so related to one another, that two different triangles can be obtained from them, that a single triangle can be obtained from them, and that no triangle can be obtained from them.

Suppose A to be less than 90°.

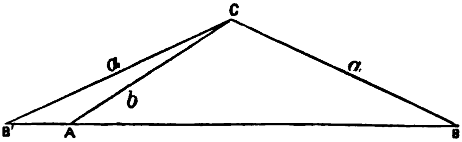
(1.) Make A (Fig. 49) equal to the given angle, and b equal to one of the given sides. From C as a centre, with a (the other given side) as a radius, describe an arc. If a is sufficiently large, and at the same time less than b, our arc will cut AB in two points, B and B′, both on the same side of A, and we shall have two triangles, ABC, AB′C, each of which will satisfy the conditions.

FIG. 49.



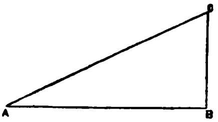
(2.) If a is equal to or greater than b, there will be only one triangle which will satisfy the conditions (*v.* Fig. 50).

FIG. 50.



(3.) If a is just equal to the perpendicular let fall from C to AB (Fig. 51), then the arc will cut the line AB in only one point, B, and we shall have only one triangle, namely, the right triangle ABC.

FIG. 51.



(4.) If a is less than BC (Fig. 50), then the arc does not cut the line AB, and there is therefore no triangle which will satisfy the conditions.

The length of the perpendicular BC (*v.* Fig. 51) is evidently $b \sin A$.

The student will note that when there are two solutions (*v*. Fig. 49), we have, —

$$CB'A = 180° - BB'C = 180° - CBA;$$

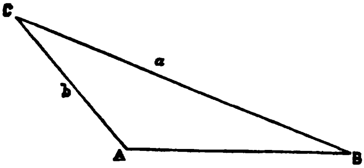
i.e., the two values of B are supplements.

When there is only one solution (Figs. 50 and 51), B cannot be greater than 90°.

Collecting the results of our discussion, we have, when A is acute, —

$$\left.\begin{array}{l}\text{Two solutions, when } a < b, \text{ and } a > b \sin A,\\ \text{One solution, when } a = b, \text{ or } a > b, \text{ or } a = b \sin A,\\ \text{No solution, when } a < b \sin A.\end{array}\right\} \quad [85]$$

FIG. 52.



When A = 90°, or A > 90°, we can never have two solutions, as can readily be seen from Fig. 52; and there will be no solution when

$$a < b, \text{ or } a = b.$$

In the present case, we have given, as has already been stated, a, b, and A.

By [67], —

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

$$\therefore \sin B = \frac{b \sin A}{a}$$

$$\therefore \log \sin B = \log b + \log \sin A + \text{colog } a,$$

$$C = 180° - (A + B).$$

$$c = \frac{a \sin C}{\sin A} \quad \therefore \log c = \log a + \log \sin C + \log \csc A.$$

Example I. Given $A = 32° \, 43'$, $a = 988.1$, $b = 672.3$. A is acute, and $a > b$; therefore, by [85], we shall have only one solution, and B must be acute.

$$\begin{array}{rl@{\qquad\qquad}rl}
\log b = & 2.8276 & \log a = & 2.9948 \\
\log \sin A = & 9.7328 & \log \sin C = & 9.9096 \\
\text{colog } a = & \underline{7.0052} & \log \csc A = & \underline{0.2672} \\
\log \sin B = & 9.5656 & \log c = & 3.1716 \\
 & & c = & 1484
\end{array}$$

$$B = 21° \, 35', \; 180° - (A + B) = C = 125° \, 42'$$

Example II. Given $A = 48° \, 34$, $a = 46.24$, $b = 60.02$. A is acute, $a < b$, and $a > b \sin A$; therefore, by [85], we shall have two solutions; and the two values of B (B and B′) will be supplements of each other.

$$\begin{array}{rl@{\qquad\qquad}rl}
\text{colog } a = & 8.3350 & B = & 76° \, 43' \\
\log b = & 1.7783 & B' = & 103° \, 17 \\
\log \sin A = & \underline{9.8749} & C = & 54° \, 43' \\
\log \sin B = & 9.9882 & C' = & 28° \, 9'
\end{array}$$

$$\begin{array}{rl@{\qquad\qquad}rl}
\log a = & 1.6650 & \log a = & 1.6650 \\
\log \sin C = & 9.9119 & \log \sin C' = & 9.6738 \\
\log \csc A = & \underline{0.1251} & \log \csc A = & \underline{0.1251} \\
\log c = & 1.7020 & \log c' = & 1.4639 \\
c = & 50.35 & c' = & 29.10
\end{array}$$

Example III.

Given $a = 49.00$, $c = 31.24$, $C = 32° \, 18'$.

Answers.

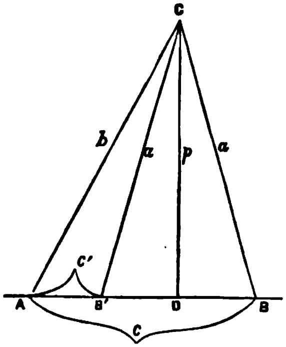
$A = 56° \, 55'$, $B = 90° \, 47'$, $b = 58 \; 47$.

$A' = 123° \, 05'$, $B' = 24° \, 37'$, $b' = 24.36$.

§ **86.** CASE II. *Second method.*

Given a, b *and* A; *find* c, B *and* C.

FIG. 53.



When a is only a little larger than p ($b \sin A$), the angles B and B′ will differ little from 90°, and cannot, therefore, be determined with accuracy from their sines (*v.* § 62). In this case, the following method will give more accurate results: b and A being given, we can, by § 69, solve the right triangle ADC (Fig. 53), and find AD and p. Now, a and p being known, we can, by § 68, solve the triangle BDC, or its equal, B′DC, and find the angles B and BCD [= B′CD], and the side DB.

From the above, we can find the required parts of the triangles ABC and AB′C, for

$$c = \text{AD} + \text{DB}, \quad c' = \text{AD} - \text{B}'\text{D},$$
$$C = \text{ACD} + \text{DCB} = 90^\circ - A + \text{DCB},$$
$$C' = \text{ACD} - \text{DCB}\ [= \text{B}'\text{CD}]$$

As has been stated before, when B and B′ fall on opposite sides of the point A, there will be only one solution; i.e., when $\text{B}'\text{D} > \text{AD}$, or when c' is negative.

Example. Given $A = 85^\circ\ 2'$, $a = 1076$, $b = 1078$. A is acute, $a < b$, and $a > b \sin A$: therefore, by [85], there are two solutions: —

1st, Solve, by § 69, the right triangle ADC (Fig. 53)

$$\begin{aligned} \log \sin A &= 9.9984 & \log \cos A &= 8.9374 \\ \log b &= \underline{3.0327} & \log b &= \underline{3.0327} \\ \log p &= 3.0311 & \log \text{AD} &= 1.9701 \\ p &= 1074 & \text{AD} &= 93.35 \end{aligned}$$

$$\text{ACD} = 90 - A = 4^\circ\ 58'$$

2d, Solve, by § 68, the right triangle BDC.

Substituting, in [66], we get, —

$$\sin\tfrac{1}{2}\,\text{DCB}=\sqrt{\frac{a-p}{2a}}\qquad \log(a-p)=0.3010$$

$$a-p=2.\qquad \log(a+p)=3.3324$$

$$a+p=2150.\qquad 2\,\boxed{3.6334}$$

$$2a=2152.\qquad \log\text{BD}=1.8167$$

$$\text{BD}=65.57$$

$$\log(a-p)=0.3010$$

$$\text{colog}\,2a=6.6671$$

$$2\,\boxed{16.9681}$$

$$\log\sin\tfrac{1}{2}\,\text{DCB}=8.4840$$

$$\tfrac{1}{2}\,\text{DCB}=1^\circ\ 45'$$

$$\text{DCB}=3^\circ\ 30'=\text{DCB}'$$

From the above we get —

$$\text{ACB}'=1^\circ\ 28'\qquad c'=27.77$$

$$\text{ACB}\ =8^\circ\ 28'\qquad c\ =158.9$$

§ **87.** CASE III. *Given two sides and the included angle a, b, and* C.

$$\text{A}+\text{B}=180^\circ-\text{C}.$$

By [68] we have

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(\text{A}+\text{B})}{\tan\frac{1}{2}(\text{A}-\text{B})},$$

$$\therefore\ (a+b)\tan\tfrac{1}{2}(\text{A}-\text{B})=(a-b)\tan\tfrac{1}{2}(\text{A}+\text{B}),$$

$$\therefore\ \tan\tfrac{1}{2}(\text{A}-\text{B})=\frac{a-b}{a+b}\tan\tfrac{1}{2}(\text{A}+\text{B}),$$

$$\therefore\ \log\tan\tfrac{1}{2}(\text{A}-\text{B})=\log(a-b)+\log\tan\tfrac{1}{2}(\text{A}+\text{B})$$
$$+\ \text{colog}\,(a+b).$$

We now have $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$.

But $\frac{1}{2}(A-B)+\frac{1}{2}(A+B)=A$,

and $\frac{1}{2}(A+B)-\frac{1}{2}(A-B)=B$.

We now have to obtain c, which we do as in Cases I. and II.

By [67], —

$$\frac{a}{\sin A}=\frac{c}{\sin C}, \quad \therefore c=\frac{a \sin C}{\sin A},$$

$$\therefore \log c=\log a+\log \sin C+\log \csc A.$$

Example I.

Given $a=48.72$, $b=3.984$, $C=68^\circ\ 34'$.

$$A+B=180-C=111^\circ\ 26',$$

$$\tfrac{1}{2}(A+B)=55^\circ\ 43'.$$

$$\begin{array}{rr}
& a+b=52.704\ \ \text{colog}\,(a+b)= \ \ 8.2782 \\
& a-b=44.736\ \ \ \log(a-b)= \ \ 1.6506 \\
& \log\tan\tfrac{1}{2}(A+B)= \ \ \underline{0.1664} \\
\log a=1.6877 & \log\tan\tfrac{1}{2}(A-B)= \ \ 0.0952 \\
\log\sin C=9.9689 & \tfrac{1}{2}(A-B)= \ \ 51^\circ\ 14' \\
\log\csc A=\underline{0.0193} & \tfrac{1}{2}(A+B)= \ \ \underline{55^\circ\ 43'} \\
\log c=1.6759 & A=106^\circ\ 57' \\
c=\ 47.41\ (v.\ \text{note}). & B=\ \ 4^\circ\ 29'
\end{array}$$

Example II. Given $a=7.942$, $b=9.998$, $C=72^\circ\ 9'$.

$$A+B=180^\circ-C=107^\circ\ 51',$$

$$\tfrac{1}{2}(A+B)=53^\circ\ 55'.5.$$

NOTE. — c might have been obtained directly from the given parts by the formula $c^2=a^2+b^2-2ab\cos C$; but the above method is to be preferred, since the formula there used is adapted to logarithmic computation.

$$\begin{array}{rl}
 & a+b = 17.94 \quad \text{colog}\,(a+b) = 8.7461 \\
 & a-b = -2.056 \quad \log\,(a-b) = 0.3131\,n \\
\log a = 0.8999 & \log \tan \tfrac{1}{2}\,(A+B) = \underline{0.1376} \\
\log \sin C = 9.9786 & \log \tan \tfrac{1}{2}\,(A-B) = 9.1968\,n \\
\log \csc A = \underline{0.1507} & \tfrac{1}{2}\,(A-B) = -8^\circ\ 56'.8 \\
\log c = 1.0292 & \tfrac{1}{2}\,(A+B) = 53^\circ\ 55'.5 \\
c = 10.70 & A = 44^\circ\ 58'.7 \\
 & B = 62^\circ\ 52'.3
\end{array}$$

In solving a triangle like the above, negative quantities can be avoided, if use is made of the following form of [68], —

$$\frac{b+a}{b-a} = \frac{\tan \frac{1}{2}\,(B+A)}{\tan \frac{1}{2}\,(B-A)}.$$

Example III.

Given $a = 6.239$, $b = 2.348$, $C = 110^\circ\ 32'$.

Answers. $A = 52^\circ\ 10'$, $B = 17^\circ\ 18'$, $c = 7.398$.

§ **88.** Case IV. *Given the three sides a, b, and c.*

$$s = \tfrac{1}{2}\,(a+b+c).$$

We can use here either [72], [73], [74], or [81]; but [74] or [81] are to be preferred, since, by their use, accurate results can be obtained for all values of the angles sought ($v.$ § 62).

We will first use [74] which is, —

$$\tan \tfrac{1}{2}\,A = +\sqrt{\frac{(s-b)(s-c)}{s\,(s-a)}},$$

$$\tan \tfrac{1}{2}\,B = +\sqrt{\frac{(s-c)(s-a)}{s\,(s-b)}},$$

$$\tan \tfrac{1}{2}\,C = +\sqrt{\frac{(s-a)(s-b)}{s\,(s-c)}}:$$

$$\begin{aligned}\therefore \log\tan\tfrac{1}{2}A &= \tfrac{1}{2}[\log(s-b)+\log(s-c)\\ &\qquad +\operatorname{colog} s+\operatorname{colog}(s-a)].\\ \log\tan\tfrac{1}{2}B &= \tfrac{1}{2}[\log(s-c)+\log(s-a)\\ &\qquad +\operatorname{colog} s+\operatorname{colog}(s-b)].\\ \log\tan\tfrac{1}{2}C &= \tfrac{1}{2}[\log(s-a)+\log(s-b)\\ &\qquad +\operatorname{colog} s+\operatorname{colog}(s-c)].\end{aligned}$$

Any two angles having been found, it is evident that the third can be found from the equation,

$$A+B+C=180^\circ.$$

It is better, however, to find each angle independently, and then to use the above equation to test the accuracy of the work.

Example. Given $a=0.0886$, $b=0.1642$, $c=0.1022$.

$$\begin{array}{rlrl} 2s= & 0.3550 & & \\ s= & 0.1775 & \log s= & \bar{1}.2492\\ s-a= & 0.0889 & \log(s-a)= & \bar{2}.9489\\ s-b= & 0.0133 & \log(s-b)= & \bar{2}.1239\\ s-c= & 0.0753 & \log(s-c)= & \bar{2}.8768 \end{array}$$

$$\begin{array}{rlrl} \log(s-b)= & \bar{2}.1239 & \log(s-c)= & \bar{2}.8768\\ \log(s-c)= & \bar{2}.8768 & \log(s-a)= & \bar{2}.9489\\ \operatorname{colog} s= & 10.7508 & \operatorname{colog} s= & 10.7508\\ \operatorname{colog}(s-a)= & 11.0511 & \operatorname{colog}(s-b)= & 11.8761\\ 2\,\big| & \overline{18.8026} & 2\,\big| & \overline{20.4526}\\ \log\tan\tfrac{1}{2}A= & 9.4013 & \log\tan\tfrac{1}{2}B= & 0.2263\\ \tfrac{1}{2}A= & 14^\circ\ 9' & \tfrac{1}{2}B= & 59^\circ\ 18\\ A= & 28^\circ\ 18' & B= & 118^\circ\ 36' \end{array}$$

$$\begin{aligned}
\log(s-a) &= \bar{2}.9489\\
\log(s-b) &= \bar{2}.1239\\
\operatorname{colog} s &= 10.7508\\
\operatorname{colog}(s-c) &= 11.1232\\
2\,&\left|\overline{18.9468}\right.\\
\log\tan\tfrac{1}{2}C &= \left|\underline{9.4734}\right.\\
\tfrac{1}{2}C &= 16^\circ\ 34'\\
C &= 33^\circ\ 8'.
\end{aligned}$$

Check. $A+B+C=180^\circ\ 02'$ (*v.* note).

§ **89.** We will now solve the problem by [81], which is well adapted to logarithmic computation when all the angles are required.

By [81], —

$$\begin{aligned}
\log\tan\tfrac{1}{2}A &= \log r + \operatorname{colog}(s-a),\\
\log\tan\tfrac{1}{2}B &= \log r + \operatorname{colog}(s-b),\\
\log\tan\tfrac{1}{2}C &= \log r + \operatorname{colog}(s-c).
\end{aligned}$$

By [79], —

$$\log r = \tfrac{1}{2}[\log(s-a)+\log(s-b)+\log(s-c)+\operatorname{colog} s].$$

Example I. Given $a=409$, $b=241$, $c=182$.

$$\begin{aligned}
s &= 416 & \operatorname{colog} s &= 7.3809\\
s-a &= 7 & \log(s-a) &= 0.8451\\
s-b &= 175 & \log(s-b) &= 2.2430\\
s-c &= 234 & \log(s-c) &= 2.3692\\
& & 2\,&\left|\overline{2.8382}\right.\\
& & \log r &= 1.4191\\
\log\tan\tfrac{1}{2}A &= 0.5740 & \tfrac{1}{2}A &= 75^\circ\ 4'\\
\log\tan\tfrac{1}{2}B &= 9.1761 & \tfrac{1}{2}B &= 8^\circ\ 32'\\
\log\tan\tfrac{1}{2}C &= 9.0499 & \tfrac{1}{2}C &= 6^\circ\ 24'
\end{aligned}$$

Note. — If the error (*v.* § 62) does not exceed 2′, the work is probably correct.

$$A = 150^\circ\ 8',\ B = 17^\circ\ 4',\ C = 12^\circ\ 48'.$$

Check. $A + B + C = 180^\circ$.

The formulas used in §§ 88 and 89 were selected because they were adapted to logarithmic computation. It is *possible*, however, to solve the problem by [69].

Example II.

Given $a = 17.865$, $b = 13.349$, $c = 11.111$.

Answers.

$$A = 93^\circ\ 23',\ B = 48^\circ\ 14',\ C = 38^\circ\ 23'.$$

EXAMPLES.

1. Given $A = 37^\circ\ 12'$ $B = 123^\circ\ 7'$ $a = .0841$ solve.
2. " $A = 18^\circ\ 17'$ $B = 94^\circ\ 9'$ $b = 8400$ "
3. " $B = 160^\circ\ 19'$ $C = 10^\circ\ 0'$ $b = 9.006$ "
4. " $B = 85^\circ\ 18'$ $C = 4^\circ\ 42'$ $c = 1$ "
5. " $a = 583.1$ $b = 503.7$ $A = 78^\circ\ 13'$ "
6. " $a = .0008$ $b = .0007$ $B = 40^\circ\ 16'$ "
7. " $a = 18.91$ $b = 9.846$ $A = 9^\circ\ 19'$ "
8. " $b = 1781$ $c = 982.7$ $B = 123^\circ\ 16'$ "
9. " $b = 81.04$ $c = 98.76$ $B = 24^\circ\ 18'$ "
10. " $a = 78.91$ $b = 68.73$ $C = 98^\circ\ 17'$ "
11. " $c = 16.70$ $a = 14.39$ $B = 16^\circ\ 16'$ "
12. " $a = 17.28$ $b = 9.861$ $c = 23.01$ "
13. " $a = 8.067$ $b = 1.76$ $c = 7.001$ "
14. " $a = 8.981$ $b = 10.08$ $B = 120^\circ\ 8'$ "

15. Consider each of the following triangles, and determine whether it is possible, or impossible; if possible, whether it has one, two, or more solutions. Give in full the reasons for your conclusions.

(1.) $A=81^\circ\ 59'\ 32''$ $B=44^\circ\ 33'\ 12''$ $C=53^\circ\ 27'\ 16''$.
(2.) $A=38^\circ\ 36'$ $B=56^\circ\ 27'$ $C=84^\circ\ 12'$.
(3.) $a=121$ $b=115.36$ $B=94^\circ\ 15'$.
(4.) $A=30^\circ$ $b=175.2$ $a=150$.
(5.) $A=30^\circ$ $b=175.2$ $a=87.6$.
(6.) $a=34.27$ $b=27.34$ $c=62.18$.

16. Each of two ships, half a mile apart, finds the angles subtended by the other ship and a fort to be respectively 85° 15′ and 83° 45′: find the distance of each from the fort.

17. From a station, B, at the base of a mountain, its summit A is seen at an elevation of 60°; after walking one mile towards the summit, up a plane making an angle of 30° with the horizon, to another station, C, the angle BCA is observed to be 135°: find the height of the mountain in yards.

18. From a window in the same horizontal plane with the bottom of a steeple, the angle of elevation of the top of the steeple is 40°; and from another window, 18 feet directly above the former, the angle of elevation is 37° 30′: find the height and distance of the steeple.

19. Two men standing at the same point, C, observe the horizontal angle subtended by the line joining two inaccessible objects, A and B; they then move away, one in the direction AC to D, the other in the direction BC to E, until each observes the horizontal angle to be half what it was before: determine the distance AB when $ACB=30^\circ$, $CD=100$, $CE=200$.

20. To determine the distance between two inaccessible objects, A and B, a person measures the distance between any two points, C and D, in the same plane with A and B: at C he observes the angles DCA and ACB; and at D, the angles ADB and BDC: determine the distance AB when $CD=100$ feet, $DCA=40^\circ$, $ACB=30^\circ$, $BDC=20^\circ$, $ADB=10^\circ$.

21. Given B, a, and the area: solve the triangle.

22. The length of a road in which the ascent is 1 foot in 5, from the foot of the hill to the top is a mile and two-thirds. What will be the length of a zigzag road in which the ascent is 1 foot in 12?

23. Given $a=24$, $b=30$, $c=18$: find the area.

24. Two angles of a triangle are 10° and 45°, and the included side is 10 feet: find the area.

25. Two sides of a triangle are equal to 3 and 12, and the included angle is 30°: find the hypothenuse of an equivalent right isosceles triangle.

26. The diagonals of a quadrilateral are in length a and b, and intersect at an angle A: prove area $=\frac{1}{2}ab\sin A$.

27. Having measured a base line of 400 yards, whose upper end was 24 feet higher than the lower one, in the same vertical plane with the top of a hill, I found the angles of elevation of the top of the hill from the lower and upper ends of the base line to be 5° 17′ and 3° 17′ respectively: find the height of the hill.

28. Being at sea, we saw two headlands, of which one bore S.W. by W., and the other W. by N. The chart showed that the first headland bore S.E. from the second, and was distant from it 23.25 miles. Find our distances from both headlands. *Ans.* 18.27 miles; 32.25 miles.

29. Two ships sail from the same port, the one S.W. 30 miles, and the other S.E. by S. 40 miles. Find the bearing and distance of the second ship from the first.

Ans. S. 74° 30′ E.; 45.08 miles.

30. An observer from a ship saw two headlands; the first bearing N.E. by E., and the second N.W. After he had sailed N.N.W. 10.25 miles, the first headland bore E. by N., and the second W.N.W. Find the bearing and distance of the first headland from the second.

Ans. N. 88° 02′ E., or nearly due east; 35.25 miles.

31. An observer saw two headlands; the first bearing S.E., the second E.S.E. After sailing E. by N. 10 miles, he saw the first bearing S. by E., and the second S.E. by S. Find the bearing and distance of the first headland from the second. *Ans.* S.S.W. 6.89 miles.

32. At one station the bearing of a cloud is N.N.W., and its angle of elevation 50° 35′. At a second station, bearing from the first N. by E., and distant 1 mile, the bearing of the cloud is W. by N. Find the height of the cloud, and its distance from each station.

Ans. 7727 feet; 10002 feet; 8494.

33. In the midst of a level plain, which is crossed by a straight road, stands a tower 250 feet high. An observer at the top of the tower sees an object which moves on the road. At first it bears N.N.W., and its angle of depression is 16° 08′; five minutes later it bears E. by S., and its angle of depression is 32° 18′. Find the direction of the road, its distance from the tower, and the rate at which the object is moving.

Ans. S.E. $\frac{1}{2}$ S.; 250.9 feet; 2.575 miles per hour.

34. Given: $a = 12.34$ chains, $b = 17.97$ chains, $C = 135°.04$.

To be computed: Area = 7.832 acres = 7 acres, 3 roods, 13 rods.

35. Given: $a = 17.95$ chains, $B = 100°$, $C = 70°$.

To be computed: Area = 85.90 acres = 85 acres, 3 roods, 24 rods.

36. Given: $a = 45.56$ chains, $b = 52.98$ chains, $c = 61.22$ chains.

To be computed: Area = 117.3 acres = 117 acres, 1.2 roods.

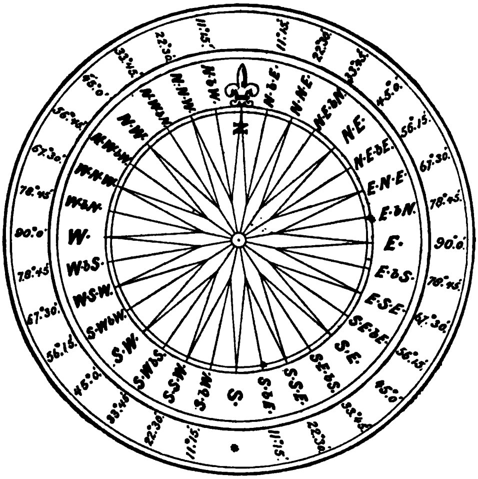
37. Given: $a = 32.56$ chains, $b = 57.84$ chains, $c = 44.44$ chains.

To be computed: Area = 71.93 acres = 71 acres, 3 roods, 29 rods.

38. Prove that the results contained in [85] § 85 are correct by the aid of the following principles: —

$$\sin B = \frac{b \sin A}{a}, \quad A + B + C = 180^\circ,$$

and, in any triangle, the greater side is opposite the greater angle.



APPENDIX I.

§ 1. The triangle of reference can be formed as stated in § 22, and also by dropping a perpendicular from any point of the terminal line on the axis Y′Y. Thus in Fig. 2_A (§ 4, Ap.), we may take OC′B′ as a triangle of reference for XOP′, and we then have $x = C'B'$, $y = OC'$, $r = OB'$; so that x is negative, and y and r positive. So for XOP‴, in the same figure, we may take $x = C'''B'''$, $y = OC'''$ $r = OB'''$.

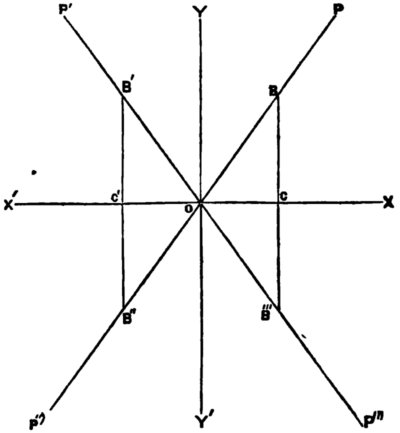
§ 2. *To find the functions of* $\varphi \pm k\,360°$, *where* k *is any integer.*

Let φ be any value of an angle XOP in any quadrant (Fig. 2_A, § 4, Ap.). If any multiple of 360° be added to or subtracted from φ, the result is also a value of the angle XOP (*v.* § 18); and the triangle of reference for φ is also a triangle of reference for $\varphi \pm k\,360°$. Hence the values of all the trigonometric functions are the same, both in numerical value and in sign, for φ and $\varphi \pm k\,360°$, that is, for *any value* of an angle XOP.

This property of the trigonometric functions is sometimes expressed by saying that they are *periodic* functions, admitting the period 360°, or 2π. We shall see presently, that the tangent and cotangent admit the simpler period, 180°, or π

§ **3.** *To find the functions of* $-\varphi$.

FIG. 1A.



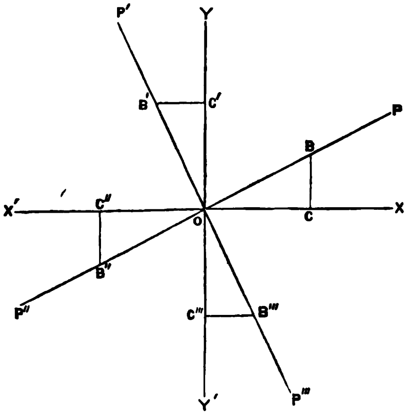
Let φ be any value of an angle in any quadrant, as XOP, XOP', XOP'', or XOP''' (Fig. 1_A). Then will $-\varphi$ be a value of the angle XOP''', XOP'', XOP', or XOP. From any point of the terminal line of φ, as B, B', B'', or B''', drop a perpendicular on the initial line, and produce it to meet the terminal line of $-\varphi$ at B''', B'', B', or B. We thus form triangles of reference for φ and $-\varphi$; and, in the two triangles thus formed, the bases and hypothenuses are equal in numerical value and in sign, while the perpendiculars are numerically equal, but opposite in sign. Hence all the trigonometric functions of φ and $-\varphi$ are numerically equal; but those functions into which the perpendicular enters are opposite in sign. That is,

$$\left.\begin{array}{ll} \sin(-\varphi) = -\sin\varphi, & \csc(-\varphi) = -\csc\varphi, \\ \cos(-\varphi) = \cos\varphi, & \sec(-\varphi) = \sec\varphi, \\ \tan(-\varphi) = -\tan\varphi, & \text{ctn}(-\varphi) = -\text{ctn}\,\varphi. \end{array}\right\} \quad [1_A]$$

§ **4.** *To find the functions of* $90° + \varphi$.

Let φ be any value of an angle in any quadrant, as XOP, XOP', XOP'', or XOP''', in Fig. 2_A, where POP', P'OP'', P″OP‴, and P‴OP are positive right angles. If 90° be added to any of these angles, it is converted into the next following angle of the series; and its triangle of reference is turned through a positive right angle (*v.* § 1, Ap.). Now, by virtue of this rotation, the base of the triangle of reference of φ becomes the perpendicular of the triangle of reference of $90° + \varphi$, and does not change its sign; but the perpendicular of the triangle of φ becomes the base of the triangle of $90° + \varphi$, and is reversed in sign; while the hypothenuse remains unchanged. That is, OC becomes OC′, C′B′ becomes C″B″, &c., CB becomes C′B′, OC′ becomes OC″, &c ; or we may write —

FIG. 2A.



$$\begin{aligned} \text{base} &= \quad \text{perp}' = -\text{base}'' = -\text{perp}''' = \text{base}^{\text{IV}}, \\ \text{perp} &= -\text{base}' = -\text{perp}'' = \quad \text{base}''' = \text{perp}^{\text{IV}}. \end{aligned}$$

Hence, for each value of φ, —

$$\sin\varphi = \frac{\text{perp}}{\text{hyp}} \text{ becomes } \frac{-\text{base}}{\text{hyp}} = -\cos(90° + \varphi);$$

$$\cos\varphi = \frac{\text{base}}{\text{hyp}} \text{ becomes } \frac{\text{perp}}{\text{hyp}} = \sin(90° + \varphi), \text{ \&c.}$$

Thus we have, —

$$\left.\begin{aligned} \sin(90°+\varphi) &= \cos\varphi, & \csc(90°+\varphi) &= \sec\varphi, \\ \cos(90°+\varphi) &= -\sin\varphi, & \sec(90°+\varphi) &= -\csc\varphi, \\ \tan(90°+\varphi) &= -\text{ctn}\,\varphi, & \text{ctn}(90°+\varphi) &= -\tan\varphi. \end{aligned}\right\} \quad [2_A]$$

§ 5. *To find the functions of* $90° - \varphi$ and $\varphi - 90°$.

Since formulas $[2_A]$ are true for *any* angle φ, we may substitute $-\varphi$ for φ in those formulas.

Thus we have, by $[1_A]$,

$$\left.\begin{aligned}
\sin(90° - \varphi) &= \cos(-\varphi) = \cos\varphi, \\
\cos(90° - \varphi) &= -\sin(-\varphi) = \sin\varphi, \\
\tan(90° - \varphi) &= -\text{ctn}(-\varphi) = \text{ctn}\,\varphi, \\
\text{ctn}(90° - \varphi) &= -\tan(-\varphi) = \tan\varphi, \\
\sec(90° - \varphi) &= -\csc(-\varphi) = \csc\varphi, \\
\csc(90° - \varphi) &= \sec(-\varphi) = \sec\varphi.
\end{aligned}\right\} \qquad [3_A]$$

Again: putting $90° - \varphi$ for φ in $[1_A]$, or $\varphi - 90°$ for $-\varphi$, we have, —

$$\left.\begin{aligned}
\sin(\varphi - 90°) &= -\cos\varphi, \\
\cos(\varphi - 90°) &= \sin\varphi, \\
\tan(\varphi - 90°) &= -\text{ctn}\,\varphi, \\
\text{ctn}(\varphi - 90°) &= -\tan\varphi, \\
\sec(\varphi - 90°) &= \csc\varphi, \\
\csc(\varphi - 90°) &= -\sec\varphi.
\end{aligned}\right\} \qquad [4_A]$$

The angles φ and $90° - \varphi$ are complements of each other: therefore, from $[3_A]$, we see that

The cos of any angle is its *complement's sin*,

The csc of any angle is its *complement's sec*,

The ctn of any angle is its *complement's tan.*

§ 6. *To find the functions of* $k\,90° \pm \varphi$, *when k is any integer, positive or negative.*

The solution of this problem requires only the *repeated* use of $[2_A]$, $[3_A]$, and $[4_A]$.

Thus, putting $90° + \varphi$ for φ, in $[2_{\text{A}}]$, we have, —

$$\left.\begin{aligned} \sin(180°+\varphi) &= \cos(90°+\varphi) = -\sin\varphi, \\ \cos(180°+\varphi) &= -\sin(90°+\varphi) = -\cos\varphi, \\ \tan(180°+\varphi) &= -\text{ctn}(90°+\varphi) = \tan\varphi. \end{aligned}\right\} \quad [5_{\text{A}}]$$

and the reciprocals of these results are the csc, sec, and ctn of $(180°+\varphi)$. We see by these formulas, that the tangent and cotangent admit the period 180°, or π (*v.* § 2, Ap.).

Putting $180°+\varphi$ for φ, in $[2_{\text{A}}]$, we have again, by $[5_{\text{A}}]$, —

$$\left.\begin{aligned} \sin(270°+\varphi) &= \cos(180°+\varphi) = -\cos\varphi, \\ \cos(270°+\varphi) &= -\sin(180°+\varphi) = \sin\varphi, \\ \tan(270°+\varphi) &= -\text{ctn}(180°+\varphi) = -\text{ctn}\,\varphi. \end{aligned}\right\} \quad [6_{\text{A}}]$$

Putting $270°+\varphi$ for φ, in $[2_{\text{A}}]$, we have the functions of $360°+\varphi$, which we have already shown, in § 2 Ap., to be equal to the functions of φ.

Putting $90°-\varphi$ for φ, in $[2_{\text{A}}]$, we have, by $[3_{\text{A}}]$, —

$$\left.\begin{aligned} \sin(180°-\varphi) &= \cos(90°-\varphi) = \sin\varphi, \\ \cos(180°-\varphi) &= -\sin(90°-\varphi) = -\cos\varphi, \\ \tan(180°-\varphi) &= -\text{ctn}(90°-\varphi) = -\tan\varphi. \end{aligned}\right\} \quad [7_{\text{A}}]$$

Putting $180°-\varphi$ for φ, in $[2_{\text{A}}]$, we have by $[7_{\text{A}}]$, —

$$\left.\begin{aligned} \sin(270°-\varphi) &= \cos(180°-\varphi) = -\cos\varphi, \\ \cos(270°-\varphi) &= -\sin(180°-\varphi) = -\sin\varphi, \\ \tan(270°-\varphi) &= -\text{ctn}(180°-\varphi) = \text{ctn}\,\varphi. \end{aligned}\right\} \quad [8_{\text{A}}]$$

Putting $270°-\varphi$ for φ, in $[2_{\text{A}}]$, we have the functions of $360°-\varphi$, which, by § 2, Ap., are equal to the functions of $-\varphi$.

Putting $\varphi - 90°$ for φ, in $[4_{\text{A}}]$, we have, —

$$\left.\begin{aligned} \sin(\varphi-180°) &= -\cos(\varphi-90°) = -\sin\varphi, \\ \cos(\varphi-180°) &= \sin(\varphi-90°) = -\cos\varphi, \\ \tan(\varphi-180°) &= -\text{ctn}(\varphi-90°) = \tan\varphi. \end{aligned}\right\} \quad [9_{\text{A}}]$$

Putting $\varphi - 180^\circ$ for φ, in [4_A], we have, by [9_A], —

$$\left.\begin{array}{l} \sin(\varphi - 270^\circ) = -\cos(\varphi - 180^\circ) = \cos\varphi, \\ \cos(\varphi - 270^\circ) = \sin(\varphi - 180^\circ) = -\sin\varphi, \\ \tan(\varphi - 270^\circ) = -\text{ctn}(\varphi - 180^\circ) = -\text{ctn}\,\varphi. \end{array}\right\} [10_A]$$

Putting $\varphi - 270^\circ$ for φ, in [4_A], we have the functions of $\varphi - 360^\circ$, which, by § 2 Ap., are equal to those of φ.

The student should now read the remarks at the end of § 33.

APPENDIX II.

COLLECTION OF FORMULAS.

AND OTHER USEFUL MATTER, FOR EASY REFERENCE.

§ 6. Circ. meas. of angle $\theta = \frac{s}{r}$.

Degree Measure. — Circular Measure.

§ 9.

$$1^\circ = \frac{\pi}{180} = 0.01745329$$

$$1' = \frac{1^\circ}{60} = 0.00029089$$

$$1'' = \frac{1'}{60} = 0.00000485$$

§ 10.

$$360^\circ = 2\pi$$

$$180^\circ = \pi$$

$$90^\circ = \tfrac{1}{2}\pi$$

$$45^\circ = \tfrac{1}{4}\pi$$

§ 9.

$$57^\circ.2957795 = \text{angular unit}$$

$$x \times 57^\circ.296.. = x$$

$$x^\circ = x \times 0.01745...$$

§ 16. XOP = XOB + BOY + YOP.

§ 18. AOB $= \varphi \pm k360^\circ$.

§ 24.

			is called the	abbreviated into
$\frac{\text{CB}}{\text{OB}}$	$\frac{y}{r}$	$\frac{\text{perpendicular}}{\text{hypothenuse}}$	sine of φ	$\sin\varphi$
$\frac{\text{OC}}{\text{OB}}$	$\frac{x}{r}$	$\frac{\text{base}}{\text{hypothenuse}}$	cosine of φ	$\cos\varphi$
$\frac{\text{CB}}{\text{OC}}$	$\frac{y}{x}$	$\frac{\text{perpendicular}}{\text{base}}$	tangent of φ	$\tan\varphi$
$\frac{\text{OC}}{\text{CB}}$	$\frac{x}{y}$	$\frac{\text{base}}{\text{perpendicular}}$	cotangent of φ	$\text{ctn}\,\varphi$
$\frac{\text{OB}}{\text{OC}}$	$\frac{r}{x}$	$\frac{\text{hypothenuse}}{\text{base}}$	secant of φ	$\sec\varphi$
$\frac{\text{OB}}{\text{CB}}$	$\frac{r}{y}$	$\frac{\text{hypothenuse}}{\text{perpendicular}}$	cosecant of φ	$\csc\varphi$
In addition to the above				
		$1 - \cos\varphi$	versed sine of φ	$\text{vers}\,\varphi$

§ 25.

φ	sin	cos	tan	ctn	sec	csc
1st quad	+	+	+	+	+	+
2d quad	+	—	—	—	—	+
3d quad	—	—	+	+	—	—
4th quad	—	+	—	—	+	—

§ 26.

$$\left.\begin{aligned} \sin\varphi &= \frac{1}{\csc\varphi}, & \cos\varphi &= \frac{1}{\sec\varphi}, & \tan\varphi &= \frac{1}{\text{ctn}\,\varphi}, \\ \csc\varphi &= \frac{1}{\sin\varphi}, & \sec\varphi &= \frac{1}{\cos\varphi}, & \text{ctn}\,\varphi &= \frac{1}{\tan\varphi}, \end{aligned}\right\} \qquad [1]$$

§ 27.

$$\frac{\sin\varphi}{\cos\varphi} = \tan\varphi \qquad [2]$$

$$\frac{\cos\varphi}{\sin\varphi} = \text{ctn}\,\varphi \qquad [3]$$

§ 28.

$$\sin^2\varphi + \cos^2\varphi = 1 \qquad [4]$$

$$\sec^2\varphi = 1 + \tan^2\varphi \qquad [5]$$

$$\csc^2\varphi = 1 + \text{ctn}^2\,\varphi \qquad [6]$$

$$\textbf{§ 30.}\quad \sin\varphi = \tfrac{1}{2}\,\text{chord}\ 2\,\varphi \qquad [7]$$

$$\textbf{§ 31.}\begin{cases}
\left.\begin{aligned}\sin(\varphi \pm k\,360^\circ) &= \sin\varphi\\ \cos(\varphi \pm k\,360^\circ) &= \cos\varphi\end{aligned}\right\} & [8]\\
\left.\begin{aligned}\sin(180^\circ - \varphi) &= \sin\varphi\\ \cos(180^\circ - \varphi) &= -\cos\varphi\end{aligned}\right\} & [9]\\
\left.\begin{aligned}\sin(180^\circ + \varphi) &= -\sin\varphi\\ \cos(180^\circ + \varphi) &= -\cos\varphi\end{aligned}\right\} & [10]\\
\left.\begin{aligned}\sin(360^\circ - \varphi) &= \sin(-\varphi) = -\sin\varphi\\ \cos(360^\circ - \varphi) &= \cos(-\varphi) = \cos\varphi\end{aligned}\right\} & [11]
\end{cases}$$

$$\textbf{§ 32.}\begin{cases}
\left.\begin{aligned}\sin(90^\circ - \varphi) &= \cos\varphi\\ \cos(90^\circ - \varphi) &= \sin\varphi\end{aligned}\right\} & [12]
\end{cases}$$

$$\textbf{§ 33.}\begin{cases}
\left.\begin{aligned}\csc(90^\circ - \varphi) &= \sec\varphi\\ \sec(90^\circ - \varphi) &= \csc\varphi\\ \tan(90^\circ - \varphi) &= \operatorname{ctn}\varphi\\ \operatorname{ctn}(90^\circ - \varphi) &= \tan\varphi\end{aligned}\right\} & [12_1]
\end{cases}$$

$$\textbf{§ 32.}\begin{cases}
\left.\begin{aligned}\sin(90^\circ + \varphi) &= \cos\varphi\\ \cos(90^\circ + \varphi) &= -\sin\varphi\end{aligned}\right\} & [13]\\
\left.\begin{aligned}\sin(270^\circ - \varphi) &= -\cos\varphi\\ \cos(270^\circ - \varphi) &= -\sin\varphi\end{aligned}\right\} & [14]\\
\left.\begin{aligned}\sin(270^\circ + \varphi) &= -\cos\varphi\\ \cos(270^\circ + \varphi) &= \sin\varphi\end{aligned}\right\} & [15]
\end{cases}$$

§ 33. Formulas 8–15 are easily remembered, if we note that when φ is coupled with 0°, 180°, or k 360°, the functions of the angles thus formed, and those of φ, are *numerically the same;* while when φ is coupled with 90°, or 270°, the functions of the angles thus formed, and those of φ, are *numerically complementary*.

The *algebraic sign* is determined in each case by supposing φ to be acute, and then noticing to what quadrant the resulting angle belongs.

§ 43. [16]

Angle	sin	cos	tan	ctn	sec	csc
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\sqrt{\frac{1}{3}}$	$\sqrt{3}$	$2\sqrt{\frac{1}{3}}$	2
45°	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{3}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\sqrt{\frac{1}{3}}$	2	$2\sqrt{\frac{1}{3}}$
90°	1	0	∞	0	∞	1
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	-2	$2\sqrt{\frac{1}{3}}$
135°	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{3}$	$-2\sqrt{\frac{1}{3}}$	2
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1
360°	0	1	0	∞	1	∞

§ 44. [17]

	sin.	cos.	tan.	ctn.	sec.	csc.
0°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$
1st qu.	inc.	dec.	inc.	dec.	inc.	dec.
90°	$+1$	± 0	$\pm\infty$	± 0	$\pm\infty$	$+1$
2d qu.	dec.	dec.	inc.	dec.	inc.	inc.
180°	± 0	-1	∓ 0	$\mp\infty$	-1	$\pm\infty$
3d qu.	dec.	inc.	inc.	dec.	dec.	inc.
270°	-1	∓ 0	$\pm\infty$	± 0	$\mp\infty$	-1
4th qu.	inc.	inc.	inc.	dec.	dec.	dec.
360°	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$

§ 46

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad [18]$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad [19]$$

§ 48.

$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}\qquad[20]$$

$$\text{ctn}(\alpha\pm\beta)=\frac{\text{ctn}\,\alpha\,\text{ctn}\,\beta\mp1}{\text{ctn}\,\beta\pm\text{ctn}\,\alpha}\qquad[21]$$

§ 49.

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\frac{\text{ctn}\,\beta+\text{ctn}\,\alpha}{\text{ctn}\,\beta-\text{ctn}\,\alpha}\qquad[22]$$

$$\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\text{ctn}\,\beta-\tan\alpha}{\text{ctn}\,\beta+\tan\alpha}\qquad[23]$$

$$\left.\begin{aligned}\sin(\alpha+\beta)\sin(\alpha-\beta)&=\sin^2\alpha-\sin^2\beta\\&=\cos^2\beta-\cos^2\alpha\end{aligned}\right\}\qquad[24]$$

$$\left.\begin{aligned}\cos(\alpha+\beta)\cos(\alpha-\beta)&=\cos^2\alpha-\sin^2\beta\\&=\cos^2\beta-\sin^2\alpha\end{aligned}\right\}\qquad[25]$$

§ 50.

$$\sin2\alpha=2\sin\alpha\cos\alpha\qquad[26]$$

$$\left.\begin{aligned}\cos2\alpha&=\cos^2\alpha-\sin^2\alpha&&\text{(I.)}\\&=2\cos^2\alpha-1&&\text{(II.)}\\&=1-2\sin^2\alpha&&\text{(III.)}\end{aligned}\right\}\qquad[27]$$

$$\tan2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}\qquad[28]$$

$$\text{ctn}\,2\alpha=\frac{\text{ctn}^2\alpha-1}{2\,\text{ctn}\,\alpha}\qquad[29]$$

§ 51.

$$\sin\frac{\alpha}{2}=\sqrt{\tfrac{1}{2}(1-\cos\alpha)}\qquad[30]$$

$$\cos\frac{\alpha}{2}=\sqrt{\tfrac{1}{2}(1+\cos\alpha)}\qquad[31]$$

$$\tan\frac{\alpha}{2}=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\qquad[32]$$

$$\text{ctn}\frac{\alpha}{2}=\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}\qquad[33]$$

§ 52.

$$\operatorname{ctn} 2\alpha = \tfrac{1}{2}(\operatorname{ctn}\alpha - \tan\alpha) \tag*{[34]}$$

$$\csc 2\alpha = \tfrac{1}{2}(\tan\alpha + \operatorname{ctn}\alpha) \tag*{[35]}$$

$$\tan\tfrac{1}{2}\alpha = \csc\alpha - \operatorname{ctn}\alpha \tag*{[36]}$$

$$\operatorname{ctn}\tfrac{1}{2}\alpha = \csc\alpha + \operatorname{ctn}\alpha = \frac{1}{\csc\alpha - \operatorname{ctn}\alpha} \tag*{[37]}$$

$$\tan(45^\circ + \tfrac{1}{2}\alpha) = \sec\alpha + \tan\alpha. \tag*{[38]}$$

$$\left.\begin{aligned}\tan(45^\circ - \tfrac{1}{2}\alpha) &= \operatorname{ctn}(45^\circ + \tfrac{1}{2}\alpha)\\ &= \sec\alpha - \tan\alpha = \frac{1}{\sec\alpha + \tan\alpha}\end{aligned}\right\} \tag*{[39]}$$

§ 53.

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta \tag*{[40]}$$

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta \tag*{[41]}$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta \tag*{[42]}$$

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta \tag*{[43]}$$

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \tag*{[44]}$$

$$\sin\alpha - \sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta) \tag*{[45]}$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \tag*{[46]}$$

$$\cos\beta - \cos\alpha = 2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta) \tag*{[47]}$$

$$\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)} \tag*{[48]}$$

$$\frac{\cos\alpha - \cos\beta}{\cos\alpha + \cos\beta} = -\tan\tfrac{1}{2}(\alpha+\beta)\tan\tfrac{1}{2}(\alpha-\beta) \tag*{[49]}$$

$$\frac{\sin\alpha \pm \sin\beta}{\cos\alpha + \cos\beta} = \tan\tfrac{1}{2}(\alpha \pm \beta) \tag*{[50]}$$

$$\frac{\sin\alpha \mp \sin\beta}{\cos\beta - \cos\alpha} = \operatorname{ctn}\tfrac{1}{2}(\alpha \pm \beta) \tag*{[51]}$$

§ 54 $\sin\alpha < \alpha < \tan\alpha$ [52]

§ 55. $\cos\alpha > 1 - \tfrac{1}{2}\alpha^2$ [59]

§ 56.

$$\left\{\begin{aligned}\sin\alpha &> \alpha - \tfrac{1}{6}\alpha^3\\ \cos\alpha &< 1 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{24}\alpha^4\end{aligned}\right. \tag*{[63]}$$

§ 65. $\left\{\begin{array}{l} b=\sqrt{(c+a)\;(c-a)} \\ a=\sqrt{(c+b)\;(c-b)} \end{array}\right\}$ [65]

§ 68. $\sin\frac{1}{2}A=\sqrt{\frac{c-b}{2c}}$, $\sin\frac{1}{2}B=\sqrt{\frac{c-a}{2c}}$ [66]

§ 71. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ [67]

§ 72. $\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$ [68]

§ 73. * $a^2=b^2+c^2-2bc\cos A$ [69]
$d^2=m^2+n^2+2mn\cos w$ [70]

§ 74. * $a=b\cos C+c\cos B$ [71]

§ 75. * $\sin\frac{1}{2}A=+\sqrt{\frac{(s-b)(s-c)}{bc}}$ [72]
* $\cos\frac{1}{2}A=+\sqrt{\frac{s(s-a)}{bc}}$ [73]

§ 76. * $\tan\frac{1}{2}A=+\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ [74]

§ 77. * $K=\frac{1}{2}bc\sin A$ [75]
* $K=\frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)}$ [76]
$K=\sqrt{s(s-a)(s-b)(s-c)}$ [77]

§ 78. $K=sr$ [78]

§ 79. $r=\frac{K}{s}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ [79]
$r=s\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C$ [80]

§ 80. * $\tan\frac{1}{2}A=\frac{r}{s-a}$ [81]

§ 81. $$^*p_a = \frac{a \sin B \sin C}{\sin A} = \frac{2K}{a}$$ [82]

§ 82. $$\begin{cases} ^*R = \frac{1}{2} a \csc A & [83] \\ R = \dfrac{abc}{4K} & [84] \end{cases}$$

§ 85.

Given A, a, b

$A < 90°$

Two solutions when $a < b$, and $a > b \sin A$

One solution when $a = b$, or $a > b$, or $a = b \sin A$

No solution when $a < b \sin A$

$A = 90°$, or $A > 90°$

Two solutions never

No solution when $a < b$, or $a = b$

---

* *v.* note, p. 80.

ANSWERS.

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25. $\text{Sin}\ A=\sqrt{\dfrac{m^2-n^2}{1-n^2}}$, $\cos B=\dfrac{n}{m}\sqrt{\dfrac{1-m^2}{1-n^2}}$.

27. $2\sin\theta\cos\theta=0$; $\therefore\ \theta=0^\circ$ or 90°.

Page 41, Chap. II.

28. 60°. *29.* 60°. *31.* -3.078. *32.* -0.875. *35.* ∞.

Page 51, Chap. III.

2. $\text{Sin}\ 15^\circ=\dfrac{\sqrt{2-\sqrt{3}}}{2}$, $\cos 15^\circ=\dfrac{\sqrt{2+\sqrt{3}}}{2}$.

6. $3\sin A-4\sin^3 A$. *7.* $4\cos^3 A-3\cos A$.

9. $\text{Sin}\ \frac{1}{2}A=\sqrt{\dfrac{5-\sqrt{21}}{10}}$, $\cos\frac{1}{2}A=\sqrt{\dfrac{5+\sqrt{21}}{10}}$.

10. 0°. *11.* 30°. *13.* $-2m\sqrt{1-m^2}$.

14. $\text{Sin}\ 2\theta=-\frac{1}{2}$; $\therefore\ \theta=-15^\circ$. *15.* $\theta=30^\circ$.

Page 52, Chap. III.

25. 120°. *26.* 45°. *28.* a.

Page 53, Chap. III.

40. $\text{Sin}\ \frac{1}{2}A=-\sqrt{\frac{1}{2}\left(1+\sqrt{1-\sin^2 A}\right)}$.

41. $-\frac{1}{2}\sqrt{2-\sqrt{3}}$. *42.* $-\sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}}$.

43. $+90^\circ$ and -90°.

PAGE 69, CHAP. V.

1. $B = 47^\circ\ 32'$, $a = 0.5688$, $b = 0.6213$.
2. $A = 38^\circ\ 35'$, $a = 0.2804$, $b = 0.3515$.
3. $B = 67^\circ\ 01'$, $b = 2321$, $c = 2521$.
4. $B = 10^\circ\ 47'$, $a = 35.34$, $c = 35.98$.
5. $A = 3^\circ\ 42'$, $b = 479.4$, $c = 480.4$.
6. $A = 67^\circ\ 01'$, $a = 2321$, $c = 2521$.
7. $A = 41^\circ\ 06'$, $B = 48^\circ\ 54'$, $a = 48.00$.
8. $A = 27^\circ\ 15'$, $B = 62^\circ\ 45'$, $a = 4.123$.
9. $A = 83^\circ\ 44'$, $B = 6^\circ\ 16'$, $b = 109.7$.
10. $A = 11^\circ\ 24'$, $B = 78^\circ\ 36'$, $b = 84.30$.
11. $A = 5^\circ\ 20'.6$, $B = 84^\circ\ 39'.4$, $c = 1.074$.
12. $A = 50^\circ\ 32'$, $B = 39^\circ\ 28'$, $c = 352.4$.
13. 36°, 54°, 1.191, 0.9634.
14. $61^\circ\ 03'$, $28^\circ\ 57'$, $\frac{\sqrt{15}}{8}$.
15. $\text{Sin } B = \cos A = \frac{p}{a}$, $b = \frac{a\,p}{\sqrt{a^2 - p^2}}$, $c = \frac{a^2}{\sqrt{a^2 - p^2}}$.
18. 100 ft., 70.7 ft.; from each other 36.6 ft.

PAGE 70, CHAP. V.

20. 88.22 yds.
22. 150 ft.
24. $a = \sqrt{2\ K \tan A}$, $b = \sqrt{\frac{2\ K}{\tan A}}$, $c = \sqrt{\frac{4\ K}{\sin 2\ A}}$.
$a = 15.86$, $b = 19.98$, $c = 25.51$.
(K stands for the area.)
26. 236.6. ft.

PAGE 92, CHAP. VI.

1. $C = 19^\circ\ 41'$, $b = 0.1166$, $c = 0.04686$.
2. $C = 67^\circ\ 34'$, $a = 2643$, $c = 7784$.
3. $A = 9^\circ\ 41'$, $a = 4.497$, $c = 4.643$.
4. $A = 90^\circ\ 00'$, $a = 12.20$, $b = 12.16$.

5. $B = 57^\circ\ 44'$, $C = 44^\circ\ 03'$, $c = 414.2$.
6. $A = 47^\circ\ 38'$, $C = 92^\circ\ 06'$, $c = 0.001082$;
or $A = 132^\circ\ 22'$, $C = 7^\circ\ 22'$, $c = 0.0001389$.
7. $B = 4^\circ\ 50'$, $C = 165^\circ\ 51'$, $c = 28.56$.
8. $C = 27^\circ\ 29'$, $A = 29^\circ\ 15,'$ $a = 1041$.
9. $C = 30^\circ\ 05'$, $A = 125^\circ\ 37'$, $a = 160.2$;
or $C = 149^\circ\ 55'$, $A = 5^\circ\ 47'$, $a = 19.85$.
10. $A = 44^\circ\ 16'.5$, $B = 37^\circ\ 26'.5$, $c = 111.9$.
11. $C = 109^\circ\ 20'$, $A = 54^\circ\ 24'$, $b = 4.957$.
12. $A = 43^\circ\ 42'$, $B = 23^\circ\ 14'$, $C = 113^\circ\ 04'$.
13. $A = 121^\circ\ 43'$, $B = 10^\circ\ 41'$, $C = 47^\circ\ 34'$.
14. $A = 50^\circ\ 25'$, $C = 9^\circ\ 27'$, $c = 1.913$.

Page 93, Chap. VI.

16. 2.605 miles, 2.612 miles. *17.* 4164.
18. 210.5 ft., 250.8 ft. *19.* 123.9.
20. 50.08 ft. when C is next A, and 29.13 ft. when it is next B.

Page 94, Chap. VI.

21. $c = \frac{2K}{a \sin B}$; the other parts by Case III.
22. 3.936 miles.
23. 216. *24.* 7.49. *25.* 6. *27.* 244.8 ft.

SPHERICAL TRIGONOMETRY.

PREFACE.

I HAVE endeavored to prepare a text-book of Spherical Trigonometry for the use of schools and colleges which, while brief and simple, shall yet be thorough, and suggestive both of the theoretical and of the practical bearings of the subject.

I have given such applications to Geometry and Astronomy, and such problems involving these applications, as will interest the student and show him that Spherical Trigonometry is not a mere mass of meaningless formulas, but an easy means of solving many practical problems of great importance.

H. N. W.

CAMBRIDGE, MASS., September 15, 1878.

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INTRODUCTION.

§ 1. Spherical Trigonometry is the application of Goniometry or Angular Analysis to Spherical Geometry; it treats principally of the solution of spherical triangles.

§ 2. We will first recall a few principles of Geometry which will be needed in our discussion.

a. A line which is perpendicular to a plane is perpendicular to every line through its foot in the plane.

b. A plane which is perpendicular to a line is perpendicular to every plane which contains the line.

c. If two planes are each perpendicular to a third plane, their line of intersection is also perpendicular to the third plane.

d. The measure of the angle between two planes is the angle between two straight lines drawn one in each plane, and perpendicular to the intersection of the planes at the same point.

e. The angle between two arcs of great circles on the same sphere is measured by the angle between their planes, or by the angle between the tangents to the arcs at their point of intersection, or by the arc of a great circle described from the vertex of the given angle as a pole and intercepted between its sides.

f. The *sides* and *angles* of any spherical triangle are respectively the supplements of the *angles* and *sides* of its polar triangle.

g. When in any spherical triangle two angles are equal, the sides opposite these angles are also equal. The converse of this is also true.

h. The arc of a great circle drawn through the vertex of an isosceles triangle and perpendicular to the base bisects both the base and the angle at the vertex.

In every spherical triangle:

i. The greater side is opposite the greater angle, and conversely.

j. Each side is less than the sum of the other two.

k. The sum of the sides is less than 360°.

l. The sum of the angles is greater than 180°.

m. Each angle is greater than the difference between 180° and the sum of the other two angles.

n. A side which differs more from 90° than another side is in the same quadrant as its opposite angle.

o. An angle which differs more from 90° than another angle is in the same quadrant as its opposite side.

§ **3.** The solution of all spherical triangles may be made to depend upon the solution of those each of whose sides and angles is less than 180°; we shall therefore discuss only the latter class of triangles.

§ **4.** It has been shown in spherical geometry that in general it is possible to construct a spherical triangle when any three of its parts are given (not excluding the case where the given parts are the three angles). We shall show in what follows, that in general it is also possible to solve a spherical triangle from the above data. In the selection of formulas we shall be governed by the principles laid down in Sections 63 and 64 of the Plane Trigonometry.

§ **5.** Two methods of discussing our subject are open to us: we can either first deduce formulas which are true for all spherical triangles, and then by substituting in these 90° for one of the angles obtain formulas true for spherical right triangles only; or we can first deduce formulas applicable to right triangles only, and then by the aid of these obtain the general formulas. The latter method is the simpler, and will be adopted in this work.

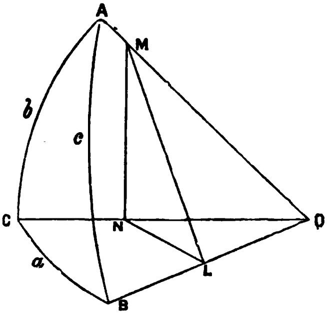
CHAPTER I.

SPHERICAL RIGHT TRIANGLES.

§ **6.** In Spherical as in Plane Trigonometry a triangle is solved by the aid of formulas expressing relations between its sides and angles, and such formulas we shall proceed to establish, limiting our investigation at present to the case of right triangles.

It is shown in spherical geometry that the sides and angles of any spherical triangle are measured by the face and diedral angles respectively of the spherical pyramid of which the triangle in question is the base, and the relations between the parts of a spherical triangle can be most easily studied by the aid of the corresponding pyramid.

FIG. I.



Suppose the spherical triangle ABC (Fig. 1) to be right angled at C; through A, B, and C draw the spherical radii AO, BO, and CO; *a*, *b*, and *c* are then the measures of the angles BOC, COA, and AOB respectively; through L, any point of BO, pass a plane perpendicular to BO, and therefore to the planes BOC and BOA (*v.* § 2 *b*); this plane LMN cuts the planes BOC and BOA in lines LN and LM, which are perpendicular to LO (*v.* § 2 *a*), and cuts the plane COA in NM; now the planes MNL and COA being each perpendicular to the plane BOC, their line of intersection NM is also perpendicular to this plane BOC (*v.* § 2 *c*), and therefore to the lines NL and NO (*v.* § 2 *a*). The angle MLN is the measure of the angle between the planes BOC and BOA (*v.* § 2 *d*), and therefore of the angle B (*v.* § 2 *e*). We have formed then the four triangles of reference:

LNM	right	angled	at N	for	B [=MLN],
OLN	"	"	" L	"	*a* [=BOC],
OLM	"	"	" L	"	*c* [=BOA],
ONM	"	"	" N	"	*b* [=COA].

We see that these triangles have certain sides in common, a property of which we shall now take advantage.

$$\cos c = \frac{\mathrm{OL}}{\mathrm{OM}} = \frac{\mathrm{ON}\cos a}{\dfrac{\mathrm{ON}}{\cos b}} = \cos a \cos b; \qquad [1]$$

$$\sin \mathrm{B} = \frac{\mathrm{NM}}{\mathrm{LM}} = \frac{\mathrm{OM}\sin b}{\mathrm{OM}\sin c} = \frac{\sin b}{\sin c}; \qquad [2]$$

$$\cos \mathrm{B} = \frac{\mathrm{LN}}{\mathrm{LM}} = \frac{\mathrm{OL}\tan a}{\mathrm{OL}\tan c} = \frac{\tan a}{\tan c}; \qquad [3]$$

$$\tan \mathrm{B} = \frac{\mathrm{NM}}{\mathrm{LN}} = \frac{\mathrm{ON}\tan b}{\mathrm{ON}\sin a} = \frac{\tan b}{\sin a}. \qquad [4]$$

It is easy to see that the corresponding formulas for A are those which we obtain by substituting in [2], [3], and [4], A and *a* in place of B and *b*, respectively.

COROLLARY I. We learn from [1] that when cos *a* and cos *b* have the same sign cos *c* is *positive*, and that when they have opposite signs cos *c* is *negative*; *c* then *is* *acute when a and b are in the same quadrant, and obtuse when they are in different quadrants.*

Corollary II. Since each part is less than 180° (§ 3) sin a is always positive; we see from [4] then that tan B and tan b always have the same sign, i. e. *an oblique angle and its opposite side are in the same quadrant.*

§ **7.** We can deduce no more formulas from our figure; we can however obtain two new relations by combining certain of the formulas already found. From [3] and [2] we have:

$$\frac{\cos B}{\sin A}=\frac{\tan a}{\tan c}\times\frac{\sin c}{\sin a}=\frac{\cos c}{\cos a}=\text{(from [1]) }\cos b;$$

$$\therefore \cos b=\frac{\cos B}{\sin A}\text{ or }\cos a=\frac{\cos A}{\sin B}. \qquad [5]$$

From [1] and [5] we have

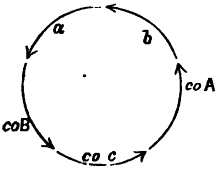
$$\cos c=\cos a\ \cos b=\frac{\cos A}{\sin B}\times\frac{\cos B}{\sin A};$$

$$\therefore \cos c=\text{ctn A ctn B}. \qquad [6]$$

Napier's Rules.

§ **8.** The formulas thus far obtained are comprised in two rules, called, from the name of their inventor, *Napier's Rules,* which, though artificial, are very generally employed as aids to the memory. In these rules the complements of the hypothenuse (c) and of the oblique angles (A) and (B) are to be used, and the right angle is to be neglected; there are then five parts which follow one another in the order indicated in the

FIG. 2.



margin. To each part there are two *adjacent* and two *opposite* parts.

The rules are as follows:

(1.) *The sine of any part is equal to the product of the (t)angents of the (a)djacent parts.*

(2.) *The sine of any part is equal to the product of the (c)osines of the (o)pposite parts.*

In the above rules the letters (*t*), (*a*), (*c*), and (*o*), were put in parenthesis in order to assist the memory; thus, (*t*) and (*a*) stand for the *(t)(a)ngents* of the *adjacent* parts, and (*c*) and (*o*) for the *(c)(o)sines* of the *opposite* parts.

The proof of the correctness of these rules lies in the fact that the formulas obtained by them are the same as those obtained in the previous sections.

§ **9.** We will now show that the six formulas which we have deduced enable us to solve all the cases of spherical right triangles which can arise; i.e. all cases in which two parts in addition to the right angle are given; let us first see how many such cases there are. Leaving out the right angle, the number of combinations that can be made out of the five remaining parts, taking two at a time, is $\frac{5\times4}{2}=10$; they are the following: (1) $a\ b$, (2) $a\ c$, (3) a A, (4) a B, (5) $b\ c$, (6) b B, (7) b A, (8) c A, (9) c B, (10) A B. These ten combinations, however, represent only six distinct cases; for (2) and (5) form only one distinct case, since in each the hypothenuse and an adjacent leg are given, and similarly from (3) and (6), (4) and (7), (8) and (9), we get only three distinct cases. We have then in all only six distinct cases, and they are the following:

Case I. Given the two legs, a and b.

Case II. Given the hypothenuse and an adjacent leg, c and a, or c and b.

Case III. Given a leg and an adjacent angle, a and B, or b and A.

Case IV. Given a leg and the opposite angle, a and A, or b and B.

Case V. Given the hypothenuse and an adjacent angle, c and A, or c and B.

Case VI. Given the two oblique angles, A and B.

§ **10.** Collecting our formulas into the following table, we see at a glance that when in addition to the right angle two parts are known, a function of each remaining part can be *directly* obtained; i. e. that each of the cases of § 9 can be solved.

Given	*Sought*
Case I. a and b.	$\cos c = \cos a \cos b$, $\tan A = \dfrac{\tan a}{\sin b}$, $\tan B = \dfrac{\tan b}{\sin a}$.
Case II. a and c.	$\cos b = \dfrac{\cos c}{\cos a}$, $\sin A = \dfrac{\sin a}{\sin c}$, $\cos B = \dfrac{\tan a}{\tan c}$.
Case III. a and B.	$\cos A = \cos a \sin B$, $\tan b = \sin a \tan B$, $\tan c = \dfrac{\tan a}{\cos B}$.
Case IV. a and A.	$\sin B = \dfrac{\cos A}{\cos a}$, $\sin b = \dfrac{\tan a}{\tan A}$, $\sin c = \dfrac{\sin a}{\sin A}$.
Case V. c and A.	$\operatorname{ctn} B = \cos c \tan A$, $\tan b = \tan c \cos A$, $\sin a = \sin c \sin A$.
Case VI. A and B.	$\cos a = \dfrac{\cos A}{\sin B}$, $\cos b = \dfrac{\cos B}{\sin A}$, $\cos c = \operatorname{ctn} A \operatorname{ctn} B$.

The computed values of the required parts should satisfy the following formulas, which may be used to test the accuracy of the solution.

$$\text{Case I.}\quad \cos c = \operatorname{ctn} A \operatorname{ctn} B.$$

$$\text{Case II.}\quad \cos b = \frac{\cos B}{\sin A}.$$

$$\text{Case III.}\quad \tan b = \tan c \cos A.$$

$$\text{Case IV.}\quad \sin b = \sin c \sin B.$$

$$\text{Case V.}\quad \tan B = \frac{\tan b}{\sin a}.$$

$$\text{Case VI.}\quad \cos c = \cos a \cos b.$$

Formulas used in this way are ordinarily called *check formulas.*

DISCUSSION OF THE DIFFERENT CASES.

§ **11.** CASE I. Given a and b. Each part being less than 180°, each required part is completely determined by the formulas given in the table.

When $a = 90°$ then $c = A = 90°$ and $B = b$.

When $a = b = 90°$ then $c = A = B = 90°$.

Example I. *Given* $a = 22° 15'$, $b = 51° 53'$.

$$\begin{array}{rrr}
\log\cos a = 9.9664 & \log\tan a = 9.6118 & \log\tan b = 0.1054 \\
\log\cos b = \underline{9.7905} & \log\sin b = \underline{9.8958} & \log\sin a = \underline{9.5783} \\
\log\cos c = 9.7569 & \log\tan A = 9.7160 & \log\tan B = 0.5271 \\
c = 55° 09' & A = 27° 28' & B = 73° 27'
\end{array}$$

$$\begin{aligned}
\textit{Check.}\quad \log\operatorname{ctn} A &= 0.2840 \\
\log\operatorname{ctn} B &= \underline{9.4729} \\
&\ 9.7569 \\
\log\cos c &= 9.7569
\end{aligned}$$

When cos c is nearly equal to unity c is small, and cannot therefore be determined with accuracy from its cosine (*v.* § 62, Plane Trig.). In this case B can be found first, and then c can be found with accuracy by the last formula of the table under CASE III.

Example II. *Given* $a = 178° 12'$, $b = 2° 25'$.

Compute $c = 176° 59'$, $A = 143° 18'$, $B = 53° 20'$.

§ **12**. CASE II. Given a and c. b and B are completely determined by the formulas in the table; but A is to be found from its sine, and corresponding to the same sine there are two supplementary angles; only one of these, however, will satisfy our triangle, namely, that one which lies in the same quadrant as a (*v.* § 6, Cor. II.).

When $a = c = 90°$ then $A = 90°$, b and B are indeterminate, i. e. there is an infinite number of triangles, each of which will satisfy the data.

When $c = 90°$ and a is not 90°, then $b = B = 90°$ and $A = a$.

The triangle is impossible when c differs more from 90° than a.

When $a = c$ and is not 90° the triangle becomes a single line equal to a or c.

Example I. *Given* $a = 132° 14'$, $c = 97° 13'$.

Compute $b = 79° 14'.5$, $A = 131° 44'$, $B = 81° 59'$.

The following example will illustrate the uncertainty which may arise when an angle near 0° or 180° is found from its cosine, and when an angle near 90° or 270° is found from its sine.

Example II. *Given* $c = 37° 42'$, $a = 37° 40'$.

$$\begin{array}{lll}
\log\cos c = 9.8983 & \log\sin a = 9.7861 & \log\tan a = 9.8876 \\
\log\cos a = \underline{9.8985} & \log\sin c = \underline{9.7864} & \log\tan c = \underline{9.8881} \\
\log\cos b = 9.9998 & \log\sin A = 9.9997 & \log\cos B = 9.9995
\end{array}$$

$$\begin{array}{rl}
\textit{Check.}\ \log\cos B = & 9.9995 \\
\log\sin A = & \underline{9.9997} \\
 & 9.9998 \\
\log\cos b = & 9.9998
\end{array}$$

We see from our four-place table that every angle between 1° 30′.3 and 1° 56′.6 has a log cos = 9.9998.*

∴ b may have any value between 1° 30′.3 and 1° 56′.6,
and likewise B " " " 2° 36′.5 " 2° 53′,
and A " " " 87° 42′ " 88° 03′.

We will now deduce three formulas which may be used to advantage when, as in the last example, accurate results cannot be obtained by the general method.

$\cos b = \frac{\cos c}{\cos a}$ or $\cos b : 1 = \cos c : \cos a$; therefore, by the theory of proportions,

$$\frac{1 - \cos b}{1 + \cos b} = \frac{\cos a - \cos c}{\cos a + \cos c};$$

∴ by [32] and [49], Plane Trig.,

$$\tan^2 \frac{b}{2} = \tan \tfrac{1}{2} (a + c) \tan \tfrac{1}{2} (c - a). \qquad [7]\, a$$

$$\sin A = \frac{\sin a}{\sin c} \text{ or } \sin A : 1 = \sin a : \sin c;$$

$$\therefore \frac{1 - \sin A}{1 + \sin A} = \frac{\sin c - \sin a}{\sin c + \sin a};$$

$$\therefore \frac{1 - \cos(90 - A)}{1 + \cos(90 - A)} = \frac{\sin c - \sin a}{\sin c + \sin a}.$$

---

* The differences between the logarithms of the cosines of angles between 1° 30′.3 and 1° 56′.6 are so small that they do not appear in a four-place table.

$\therefore$ by [32] and [48], Plane Trig.,

$$\tan^2 (45^\circ - \tfrac{1}{2} \text{A}) = \tan \tfrac{1}{2} (c - a) \operatorname{ctn} \tfrac{1}{2} (c + a). \qquad [7]\, b$$

$$\cos \text{B} = \frac{\tan a}{\tan c} \text{ or } \cos \text{B} : 1 = \tan a : \tan c;$$

$$\therefore \frac{1 - \cos \text{B}}{1 + \cos \text{B}} = \frac{\tan c - \tan a}{\tan c + \tan a} = \frac{\sin c \cos a - \cos c \sin a}{\sin c \cos a + \cos c \sin a};$$

$\therefore$ by [32] and [18], Plane Trig.,

$$\tan^2 \tfrac{1}{2} \text{B} = \sin (c - a) \div \sin (c + a). \qquad [7]\, c$$

We will now work Example II. by the new formulas just obtained.

$$\begin{array}{rl@{\qquad}rl}
c = & 37^\circ\ 42' & a = & 37^\circ\ 40' \\
c + a = & 75^\circ\ 22' & c - a = & 0^\circ\ 02' \\
\tfrac{1}{2}(c + a) = & 37^\circ\ 41' & \tfrac{1}{2}(c - a) = & 0^\circ\ 01' \\
\log \tan \tfrac{1}{2}(c + a) = & 9.8879 & \log \operatorname{ctn} \tfrac{1}{2}(c + a) = & 0.1121 \\
\log \tan \tfrac{1}{2}(c - a) = & \underline{6.4637} & \log \tan \tfrac{1}{2}(c - a) = & \underline{6.4637} \\
 & 2)\,\underline{6.3516} & & 2)\,\underline{6.5758} \\
\log \tan \tfrac{1}{2} b = & 8.1758 & \log \tan (45 - \tfrac{1}{2} \text{A}) = & 8.2879
\end{array}$$

$$\begin{array}{rl@{\qquad}rl}
\tfrac{1}{2} b = & 0^\circ\ 51'.5 & 45^\circ - \tfrac{1}{2} \text{A} = & 1^\circ\ 06'.7 \\
b = & 1^\circ\ 43' & 90^\circ - \text{A} = & 2^\circ\ 13'.4 \\
\log \sin (c - a) = & 6.7648 & \text{A} = & 87^\circ\ 46'.6 \\
\log \sin (c + a) = & \underline{9.9857} & & \textit{Check.} \\
 & 2)\,\underline{6.7791} & \log \cos \text{B} = & 9.9995 \\
\log \tan \tfrac{1}{2} \text{B} = & 8.3896 & \log \sin \text{A} = & \underline{9.9997} \\
\tfrac{1}{2} \text{B} = & 1^\circ\ 24'.3 & & 9.9998 \\
\text{B} = & 2^\circ\ 48'.6 & \log \cos b = & 9.9998
\end{array}$$

§ 13. CASE III. Given a and B. Each part is completely determined by the formulas in the table.

When $a = B = 90°$ then $A = b = c = 90°$.

Example. *Given* $a = 88° \, 05'$, $B = 38° \, 20'$.

Compute $c = 88° \, 30'$, $A = 88° \, 49'$, $b = 38° \, 19'$.

What other method for finding A may be used to advantage when cos A is nearly equal to 1?

§ 14. CASE IV. Given a and A. Each required part is to be obtained from its sine, to which may belong both an acute and an obtuse value; from this it would appear that there are *eight* different triangles corresponding to the given values of a and A; b and B must however lie in the same quadrant (§ 6, Cor. II.); and from § 6, Cor. I., we see that, a being given, c can have only one value when b is acute, and only one value also when b is obtuse; there are therefore only two possible sets of values for the required parts b, B, and c: i. e. (1) b *acute*, B *acute*, c *in same quadrant as* a; (2) b *obtuse*, B *obtuse*, c *in different quadrant from* a.

When $\sin a = \sin A$ and < 1 then $B = b = c = 90°$.

When $a = A = 90°$ then $c = 90°$, b and B are indeterminate.

The triangle is impossible when (1) a and A are in different quadrants (*v.* § 6, Cor. II.), and (2) when $\sin a > \sin A$; for here $\sin c = \frac{\sin a}{\sin A} > 1$, which is impossible.

Example I. *Given* $a = 147° \, 49'$, $A = 137° \, 36'$.

$\log \cos A = 9.8683\text{n}$	$\log \tan a = 9.7989\text{n}$	$\log \sin a = 9.7264$
$\log \cos a = 9.9275\text{n}$	$\log \tan A = 9.9605\text{n}$	$\log \sin A = 9.8288$
$\log \sin B = 9.9408$	$\log \sin b = 9.8384$	$\log \sin c = 9.8976$
$B_1 = 60° \, 45'$	$b_1 = 43° \, 34'$	$c_1 = 52° \, 11'$
$B_2 = 119° \, 15'$	$b_2 = 136° \, 25'$	$c_2 = 127° \, 49'$

Check.

$$\begin{aligned} \log\sin c &= 9.8976 \\ \log\sin \mathrm{B} &= \underline{9.9408} \\ & \ 9.8384 \\ \log\sin b &= 9.8384. \end{aligned}$$

The parts b_1, B_1, and c_2 belong to one of the possible triangles, and the parts b_2, B_2, and c_1 to the other.

When the formulas in the table (§ 10) give inaccurate results the following may be used:

$$\left.\begin{aligned} \tan^2(45^\circ - \tfrac{1}{2}\,\mathrm{B}) &= \tan\tfrac{1}{2}\,(\mathrm{A}-a)\tan\tfrac{1}{2}\,(\mathrm{A}+a). && (a) \\ \tan^2(45^\circ - \tfrac{1}{2}\,b) &= \sin(\mathrm{A}-a) \div \sin(\mathrm{A}+a). && (b) \\ \tan^2(45^\circ - \tfrac{1}{2}\,c) &= \tan\tfrac{1}{2}\,(\mathrm{A}-a)\,\mathrm{ctn}\,\tfrac{1}{2}\,(\mathrm{A}+a). && (c) \end{aligned}\right\}[8]$$

These may be deduced from the general formulas of CASE IV. by the following formulas of Plane Trigonometry:

$$(a)\ \text{from } \sin\mathrm{B} = \frac{\cos\mathrm{A}}{\cos a} \text{ by [49] and [32];}$$

$$(b)\ \text{from } \sin b = \frac{\tan a}{\tan\mathrm{A}} \text{ by [22];}$$

$$(c)\ \text{from } \sin c = \frac{\sin a}{\sin\mathrm{A}} \text{ by [48] and [32].}$$

Example II. *Given* $a = 34^\circ\ 06'$, $\mathrm{A} = 34^\circ\ 08'$.

Compute $\mathrm{B}_1 = 88^\circ\ 24'$ $b_1 = 87^\circ\ 08'$ $c_1 = 87^\circ\ 38'$
$\mathrm{B}_2 = 91^\circ\ 36'$, $b_2 = 92^\circ\ 52'$, $c_2 = 92^\circ\ 22'$.

§ 15. CASE V. Given c and A.

Each required part is completely determined by the formulas in the table except a, which must, by § 6, Cor. II, lie in the same quadrant as A.

When $c = \mathrm{A} = 90^\circ$ then $a = 90^\circ$, b and B are indeterminate.

When $c = 90^\circ$ and A is not 90° then $b = \mathrm{B} = 90^\circ$, and $a = \mathrm{A}$.

When $A = 90°$ and c is not 90° then $a = c$, and the triangle becomes a single line equal to a or c.

Example I. *Given* $c = 112° \ 48'$, $A = 56° \ 12'$.

Compute $a = 49° \ 56'$, $b = 127° \ 05'$, $B = 120° \ 04'$.

What other method for finding a may be used to advantage when $\sin a$ is nearly equal to unity?

Example II. *Given* $c = 87° \ 12'$, $A = 88° \ 12'$.

Compute $B = 32° \ 45'$, $b = 32° \ 42'$, $a = 86° \ 40'$.

§ **16.** CASE VI. Given A and B.

Each required part is completely determined by the formulas in the table (§ 10).

When $A = B$ and is not 90° then $a = b$.
When $A = B = 90°$ then $a = b = c = 90$.
When $A = 90°$ and B is not 90° then $a = c = 90°$ and $b = B$.

Example I. *Given* $A = 63° \ 15'$, $B = 135° \ 34'$.

Compute $a = 50° \ 00'$, $b = 143° \ 06'$, $c = 120° \ 56'$.

For cases of inaccuracy the following formulas may be used:

$$\left.\begin{aligned} \operatorname{ctn}^2 \tfrac{1}{2} a &= \frac{\tan \frac{1}{2}(90° + B - A)}{\tan \frac{1}{2}(B + A - 90°)}. && (a) \\ \operatorname{ctn}^2 \tfrac{1}{2} b &= \frac{\tan \frac{1}{2}(90° + A - B)}{\tan \frac{1}{2}(B + A - 90°)}. && (b) \\ \operatorname{ctn}^2 \tfrac{1}{2} c &= -\frac{\cos (B - A)}{\cos (B + A)}. && (c) \end{aligned}\right\} [9]$$

These may be deduced from the general formulas of Case VI. by the following formulas of Plane Trigonometry:

(*a*) from $\cos a = \frac{\cos A}{\sin B}$ by [48] and [33],

(*b*) from $\cos b = \frac{\cos B}{\sin A}$ by [48] and [33],

(*c*) from $\cos c = \operatorname{ctn} A \operatorname{ctn} B = \frac{\operatorname{ctn} A}{\tan B}$ by [23] and [33].

Example II. *Given* A = 92° 08′, B = 50° 02′.

Compute a = 92° 48′, b = 50° 00′, c = 91° 48′.

EXAMPLES.

	Given		*Compute*		
	a	b	c	A	B
1.	36° 27′	43° 35′	54° 21′	46° 59′	58° 02′
2.	86° 40′	32° 40′	87° 12′	88° 12′	32° 43′
3.	2° 01′	0° 27′	2° 04′	77° 26′	12° 35′
4.	120° 10′	151° 00′	63° 56′	105° 44′	147° 20′
	a	c	b	A	B
5.	14° 17′	23° 50′	19° 17′	37° 38′	54° 49′
6.	32° 09′	44° 33′	32° 42′	49° 19′	50° 19′
7.	12° 15′	95° 44′	95° 52′	12° 19′	91° 15′
8.	8° 12′	171° 00′	176° 17′	65° 45′	155° 29′
	a	B	c	b	A
9.	92° 48′	50° 02′	91° 48′	50° 00′	92° 09′
10.	96° 50′	50° 12′	94° 23′	50° 00′	95° 15′
11.	20° 20′	38° 10′	25° 14′	15° 16′	54° 35′
12.	3° 56′	85° 47′	43° 04′	42° 56′	5° 46′
	a	A	c_1	b_1	B_1
13.	33° 40′	43° 21′	53° 52′	44° 53′	60° 54′
14.	133° 19′	124° 00′	118° 39′	45° 40′	54° 36′
15.	111° 44′	95° 45′	111° 00′	14° 38′	15° 42′
16.	87° 12′	87° 52′	88° 11′	49° 34′	49° 38′

	Given		*Compute*		
	c	A	*a*	*b*	B
17.	69° 25′	54° 55′	49° 59′	56° 51′	63° 25′
18.	118° 40′	128° 00′	136° 15′	48° 24′	58° 27′
19.	58° 30′	22° 01′	18° 39′	56° 33′	78° 04′
20.	89° 30′.5	89° 16′	89° 07′	56° 17′	56° 17′
	A	B	*c*	*a*	*b*
21.	63° 15′	135° 34′	120° 56′	50° 00′	143° 06′
22.	116° 43′	116° 31′	75° 27′	120° 10′	119° 59′
23.	47° 00′	57° 59′	54° 19′	36° 27′	43° 33′
24.	10° 46′	79° 34′	14° 27′	2° 40′	14° 14′

The following triangles, although not right triangles, can be solved by principles already explained:

	Given			*Compute*		
	c	*a*	*b*	A	B	C
25.	90°	60° 04′	40° 10′	39° 19′	28° 10′	133° 01′
26.	90°	174° 13′	94° 08′	175° 57′	135° 40′	135° 35′
	a	A	C	*b*	*c*	B
27.	90°	28° 06′	16° 08′	122° 48′	36° 09′	156° 41′
	c	A	B	C	*a*	*b*
28.	90°	110° 48′	135° 36′	104° 42′	104° 53′	133° 40′
	a	*b*	B	*c*	A	C
29.	28° 00′	28° 00′	79° 00′	11° 35′	79° 00′	24° 50′
	A	B	C	*a*	*b*	*c*
30.	120° 00′	120° 00′	80° 00′	133° 28′	133° 28′	55° 34′

31. A spherical square is a spherical quadrilateral which has equal sides and equal angles. The diagonals divide it into four equal spherical right triangles. Given a side (a) of the spherical square: find an angle (A).

Ans. $\text{ctn}\,\frac{1}{2}\text{A} = \sqrt{\cos a}.$

32. A line makes with a plane an angle ϕ; through the foot of the line and in the plane draw a second line making with the projection of the first on the plane the angle α; find the angle x made by the second line with the first.

Ans. $\cos x = \cos\phi \cos\alpha$.

33. Given the number of sides of a regular spherical polygon equal to n and each angle equal to A; find a side (a) of the polygon and the polar radii of the circumscribed and inscribed circles.

$$\textit{Ans.}\ \cos\tfrac{1}{2}a = \frac{\cos\frac{\pi}{n}}{\sin\frac{1}{2}A},$$

$$\cos r = \frac{\cos\frac{1}{2}A}{\sin\frac{\pi}{n}},\quad \cos R = \text{ctn}\,\tfrac{1}{2}A\,\text{ctn}\,\frac{\pi}{n}.$$

34. Find the angles between the adjacent faces of each of the five regular polyhedrons.

Ans. Tetrahedron 70° 32′, Hexahedron 90°, Octahedron 109° 28′, Dodecahedron 116° 35′, Icosahedron 138° 15′.

35. The base of a regular pyramid is a decagon; each angle at the vertex of the pyramid is 18°; find the angle (i) which each lateral face makes with the base and the angle (ϕ) made by the lateral faces with one another.

Ans. $i = 60° 54'$, $\phi = 148° 40'$.

In Appendix I. will be found a few remarks about Astronomy, with which the student should become perfectly familiar before attempting to work the following examples.

36. From the longitude of the sun (l) and the obliquity of the ecliptic (e) find the sun's right ascension (α) and declination (δ). *Ans.* $\tan\alpha = \tan l \cos e$, $\sin\delta = \sin l \sin e$.

37. (a) From the latitude (ϕ) of a place on the earth's surface and the declination (δ) of the sun on a given day, find the times and places of the rising and setting of the sun and its distance from the zenith at noon.

(*b*) When are days and nights equal ?

(Neglect the effect of refraction.)

Ans. (*a*) $\cos\omega = \tan\phi\tan\delta$, $\cos d = \sin\delta : \cos\phi$, where ω denotes the angle, which reduced to hours and minutes is the time before and after midnight of setting and rising; d is the distance from the north point of the horizon, $\phi - \delta$ is the distance from the zenith at noon.

(*b*) Days and nights are equal when the sun rises at 6 A. M. and sets at 6 P. M., i. e. when $\omega = 90° = 6$ h.; this occurs (1) at all places on the earth when $\delta = 0$ (March 21 and September 21) and (2) at all times of the year for places on the equator ($\phi = 0$).

38. When does the solution of Example 37 become impossible ? when indeterminate ? and what follows for places so situated on the earth's surface as to give these results ?

Ans. Impossible when $90° - \phi < \delta$, i. e. when the polar distance of the place is less than the declination of the sun; in such cases the sun neither rises nor sets, but is either entirely above or entirely below the horizon during the entire day.

Indeterminate when $90° - \phi = \delta = 0$, i. e. at the poles when $\delta = 0$; here the sun is just in the horizon during the entire day.

39. When and where does the sun set in Cambridge (lat. 42° 23′) on the longest day (June 21, $\delta = +23° 27'$) ? when and where on the shortest day (December 21, $\delta = -23° 27'$) ?

Ans. $\left.\begin{aligned}\omega &= 66° 41' = 4 \text{ h. } 27 \text{ m.}\\ d &= 57° 24'\end{aligned}\right\}$ (longest).

$\left.\begin{aligned}\omega &= 113° 19' = 7 \text{ h. } 33 \text{ m.}\\ d &= 122° 36'\end{aligned}\right\}$ (shortest).

40. Solve Ex. 39, substituting New Haven (lat. $= 41° 17'$) for Cambridge.

$$\textit{Ans.}\quad \left.\begin{array}{r}\omega = 67° \ 37' = 4 \text{ h. } 30.5 \text{ m.} \\ d = 58° \ 01'.5\end{array}\right\} \text{(longest).}$$

$$\left.\begin{array}{r}\omega = 112° \ 23' = 7 \text{ h. } 29.5 \text{ m.} \\ d = 121° \ 58'.5\end{array}\right\} \text{(shortest).}$$

41. (*a*) Find the altitude (h) and azimuth (a) of the sun at a given place in the northern hemisphere (lat. ϕ and on a given day (dec. δ) at 6 A. M.

(*b*) How does the altitude change with a change of place . What is its value for a place on the equator ? what for a place at the pole ? What is the greatest possible altitude ?

(*c*) Why is the sun never above the horizon before 6 A. M. in winter ?

(*d*) How does the azimuth at the same place change during the year ? how on the same day (i. e. declination of the sun constant) when the position of the place changes ? What is the azimuth of a place on the equator ? what of a place at the pole ?

Answers. (*a*) $\sin h = \sin \phi \sin \delta$, $\text{ctn } d = \text{ctn } (180° + d) = \cos \phi \tan \delta$. $(180° + d) = a$, where $d =$ distance from the north point of the horizon.)

(*b*) In summer the altitude increases as the distance of the place from the equator increases; at the equator it is 0°; at the pole it is equal to the declination of the sun, which is the greatest possible altitude that the sun can have at this time of the day; for other places on the northern hemisphere the altitude is a maximum on the longest day (June 21, when $\delta = 23° \ 27'$).

(*c*) In winter δ, and therefore h, is negative.

(*d*) The azimuth increases with a decrease in δ and an increase in ϕ; at the equator it is $270° - \delta$, and at the pole it is 270°.

42. Find the altitude and azimuth of the sun in Cambridge ($\phi = 42° 23'$) on June 21 ($\delta = 23° 27'$) at 6 A. M.

Ans. $h = 15° 34'$. $a = 252° 10'$.

43. (*a*) At what times of the day at a given place in the northern hemisphere (i. e. ϕ given) and on a given day (i. e. δ given) is the sun exactly in the east or west? (*b*) What are the results when $\delta = 0$, i. e. when days and nights are equal? (*c*) Why in summer can the sun never be in the east before 6 A. M. and never in the west after 6 P. M.?

(*d*) What results are obtained when the latitude ϕ is less than the declination δ? what when it is equal to δ? What results are obtained for the north pole?

Ans. (*a*) $\cos \omega = \tan \delta \operatorname{ctn} \phi$, where ω reduced to hours is the time before or after noon.

(*b*) When $\delta = 0°$ then $\omega = 90° = 6$ h.

(*c*) The place being north of the equator ϕ is positive; in summer δ is positive; therefore $\cos \omega$ is positive and ω is less than 90° or 6 h.

(*d*) When $\phi < \delta$ then $\cos \omega > 1$, and the problem is impossible. When $\phi = \delta$ then $\omega = 0°$ or the sun is in the east and west at noon; this means that the sun is at noon in the zenith, and therefore in that vertical circle which passes through the east and west points of the horizon. At the pole $\cos \omega = 0$, or $\omega = 90° = 6$ h.

44. At a place on the earth's surface whose latitude is ϕ, a rod OA, pointed toward the north pole, makes with a horizontal plane an angle $\text{BOA} = \phi$; let OC be the position of the shadow of the rod at a given time of the day; find the angle BOC.

Given $\phi = 42° 23'$ (Cambridge) $t = 2 \text{ h.} = 30°$.

Ans. $\tan \text{BOC} = \sin \phi \tan t$; $\therefore \text{BOC} = 21° 16'$.

This example tells us how to construct a horizontal sundial. The rod is made to make with the horizontal plane an angle ϕ, because it is in this position parallel to the earth's polar diameter, about which as an axis the sun appears to move.

45. Through O (Ex. 44) pass a plane perpendicular to OB; extend the rod OA through this vertical plane so that its shadow may fall upon it; find the angle S which the shadow makes with the projection of the rod on the plane at a given time t (vertical sundial).

Given $\phi = 42^\circ\ 23'$ (Cambridge) $t = 3\text{ h.} = 45^\circ$.

Ans. $\tan S = \cos\phi\ \tan t$; $\therefore\ S = 36^\circ\ 27'$.

46. Find the length, in geographical miles, of the great circle arc joining Halifax — long. 63° 35′ W., lat. 44° 40′ N. — with Cape Ferret (near Bordeaux, France) — long. 1° 14′ W., lat. same as that of Halifax. *Ans.* 2592 miles.

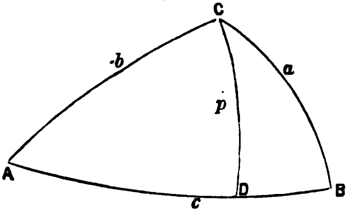
Compare Ex. **31**, p. **71**, Plane Trig.

CHAPTER II.

THE SPHERICAL TRIANGLE IN GENERAL.

§ **17.** *Theorem.* In every spherical triangle the sines of the sides are proportional to the sines of the opposite angles.

FIG. 3.



Through C (Fig. 3) draw the great circle arc p perpendicular to c; applying [2] to the two right triangles thus formed, we have $\sin p = \sin A \sin b = \sin B \sin a$;

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}, \text{ and similarly } \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c};$$

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \quad \text{Q. E. D.} \qquad [10]$$

Why is the result the same when p cuts AB produced?

§ **18.** *Problem.* To find the cosine of a side of any spherical triangle in terms of functions of the opposite angle and the other two sides.

From [1] we have in Fig. 3

$$\cos a = \cos p \cos \text{DB};$$

but $\text{DB} = \text{AB} - \text{AD}$, $\therefore \cos \text{DB} = \cos c \cos \text{AD} + \sin c \sin \text{AD}$;

and from [1] again $\cos p = \dfrac{\cos b}{\cos \text{AD}}$;

$$\therefore\ \cos a = \frac{\cos b\ \cos c\ \cos \mathrm{AD}}{\cos \mathrm{AD}} + \frac{\sin c\ \cos b\ \sin \mathrm{AD}}{\cos \mathrm{AD}},$$

and from [3] tan AD = cos A tan b;

$$\left.\begin{array}{l} \therefore\ \cos a = \cos b\ \cos c + \sin b\ \sin c\ \cos \mathrm{A}, \\ \text{and similarly} \\ \cos b = \cos c\ \cos a + \sin c\ \sin a\ \cos \mathrm{B}, \\ \cos c = \cos a\ \cos b + \sin a\ \sin b\ \cos \mathrm{C}. \end{array}\right\} [11]$$

NOTE. What has been said about the measurement of straight lines in § 2, Plane Trig., will apply equally well to the measurement of arcs; bearing this in mind, and also the fact that the sines of an arc and its supplement are the same, and that the cosines of an arc and its negative are the same, §§ 17 and 18 will be seen to apply equally well to the cases where D does not fall between A and B.

§ **19.** Let A′B′C′ be the polar triangle of ABC, then by § 2 *f*,

$$\begin{array}{ll} \mathrm{A}' = 180° - a, & a' = 180° - \mathrm{A}, \\ \mathrm{B}' = 180° - b, & b' = 180° - \mathrm{B}, \\ \mathrm{C}' = 180° - c, & c' = 180° - \mathrm{C}. \end{array}$$

Applying [11] to the triangle A′B′C′ we get

$$\cos a' = \cos b'\ \cos c' + \sin b'\ \sin c'\ \cos \mathrm{A}',$$

or substituting from above

$$-\cos \mathrm{A} = (-\cos \mathrm{B})\ (-\cos \mathrm{C}) + \sin \mathrm{B}\ \sin \mathrm{C}\ (-\cos a);$$

$$\left.\begin{array}{l} \text{or}\quad -\cos \mathrm{A} = \cos \mathrm{B}\ \cos \mathrm{C} - \sin \mathrm{B}\ \sin \mathrm{C}\ \cos a, \\ \text{and similarly} \\ -\cos \mathrm{B} = \cos \mathrm{C}\ \cos \mathrm{A} - \sin \mathrm{C}\ \sin \mathrm{A}\ \cos b, \\ \text{and} -\cos \mathrm{C} = \cos \mathrm{A}\ \cos \mathrm{B} - \sin \mathrm{A}\ \sin \mathrm{B}\ \cos c. \end{array}\right\} [12]$$

§ **20.** From [11] $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$;

$\therefore$ from Plane Trig. [31] and the above

$$\cos \tfrac{1}{2} A = \sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{\sin b \sin c + \cos a - \cos b \cos c}{2 \sin b \sin c}}$$

$$= \sqrt{\frac{\cos a - \cos (b + c)}{2 \sin b \sin c}} \text{ (v. Plane Trig. [19])}$$

$$= \sqrt{\frac{\sin \frac{1}{2} (a + b + c) \sin \frac{1}{2} (b + c - a)}{\sin b \sin c}} \text{ (v. Plane Trig. [47]).}$$

Now put $a + b + c = 2s$ and $\therefore$

$-a + b + c = 2(s - a)$ and we get

$$\left.\begin{aligned} \cos \tfrac{1}{2} A &= \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}, \\ &\text{and similarly} \\ \cos \tfrac{1}{2} B &= \sqrt{\frac{\sin s \sin (s - b)}{\sin c \sin a}}, \\ \cos \tfrac{1}{2} C &= \sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin b}}. \end{aligned}\right\} [13]$$

In a similar manner may be obtained

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} (a - b + c) \sin \frac{1}{2} (a + b - c)}{\sin b \sin c}}$$

$$= \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}; \qquad [14]$$

and from [14] and [13]

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}. \qquad [15]$$

By advancing the letters we get similar formulas for B and C. We have agreed to restrict ourselves to triangles in which each part is less than 180°, therefore $\frac{1}{2}$ A, $\frac{1}{2}$ B, and $\frac{1}{2}$ C are each less than 90° and their functions are positive; we must then give the + sign to the radicals in [13], [14], and [15].

Problem. Prove that the values of the sine, cosine, and tangent of the half angle contained in [13], [14], and [15] are real, i. e. that the quantity under the radical is in each case positive.

§ **21.** From [12] we get

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

Now proceeding as in § 20 and putting $A + B + C = 2S$ we get

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}, \qquad [16]$$

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\sin B \sin C}}, \qquad [17]$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}. \qquad [18]$$

Let the student write the corresponding formulas for *b* and *c*.

Corollary 1. The above formulas give real values for $\sin \frac{1}{2} a$, $\cos \frac{1}{2} a$, and $\tan \frac{1}{2} a$.

Proof. 1st. Sin A, sin B, and sin C are each positive, because A, B, and C are each less than 180°.

2d. $2S > 180° < 540°$ (§ 2 *l* and § 3).

$\therefore S > 90° < 270°$, i. e. cos S is negative and $-\cos S$ is positive.

3d. $S - A = \frac{1}{2}(B + C - A)$. In the polar triangle whose sides we will denote by a', b', and c', we have $a' < b' + c'$ (§ 2 *j*);

$\therefore 180 - A < 180 - B + 180 - C$; $\therefore B + C - A < 180°$ and $\frac{1}{2}(B + C - A) = S - A < 90°$; $\therefore \cos(S - A)$ is positive, and in like manner $\cos(S - B)$ and $\cos(S - C)$ are each positive.

Since then each factor of the quantities under the radical is positive the quantities themselves are positive and their roots are real.

Corollary 2. The sign of the root is in each case positive.

§ **22**. Substituting in the following formula (Plane Trig. [19]), $\cos\frac{1}{2}(A \pm B) = \cos\frac{1}{2}A \cos\frac{1}{2}B \mp \sin\frac{1}{2}A \sin\frac{1}{2}B$, the values for $\sin\frac{1}{2}A$, $\sin\frac{1}{2}B$, $\cos\frac{1}{2}A$, and $\cos\frac{1}{2}B$ obtained in § 20, we get

$$\cos\tfrac{1}{2}(A \pm B) = \frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}$$

$$\mp \frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}$$

$$= \frac{\sin s \mp \sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}.$$

Now $s = \frac{a+b}{2} + \frac{c}{2}$, and $s - c = \frac{a+b}{2} - \frac{c}{2}$;

$$\therefore \sin s - \sin(s-c) = 2\sin\tfrac{1}{2}c\cos\frac{a+b}{2} \quad (v.\ \text{Plane Trig. [45]})$$

$$\text{and } \sin s + \sin(s-c) = 2\cos\tfrac{1}{2}c\sin\frac{a+b}{2} \quad (v.\ \text{Plane Trig. [44]})$$

$$\text{and } \sin c = 2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c \quad (v.\ \text{Plane Trig. [26]})$$

$$\therefore \cos\tfrac{1}{2}(A+B)=\frac{2\sin\frac{1}{2}c\cos\frac{1}{2}(a+b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\sin\tfrac{1}{2}C$$

$$=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\tfrac{1}{2}C, \qquad [19]$$

$$\text{and } \cos\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\tfrac{1}{2}C. \qquad [20]$$

Now substituting as before in the formula

$\sin\frac{1}{2}(A\pm B)=\sin\frac{1}{2}A\cos\frac{1}{2}B\pm\cos\frac{1}{2}A\sin\frac{1}{2}B$ (Plane Trig. [18]),

we get

$$\sin\tfrac{1}{2}(A\pm B)=\frac{\sin(s-b)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

$$\pm\frac{\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

$$=\frac{\sin(s-b)\pm\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.$$

Now $s-b=\dfrac{c+a-b}{2}$, and $s-a=\dfrac{b+c-a}{2}=\dfrac{c-(a-b)}{2}$;

$\therefore \sin(s-b)+\sin(s-a)=2\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)$ (Plane Trig. [44]);

and $\sin(s-b)-\sin(s-a)=2\cos\frac{1}{2}c\sin\frac{1}{2}(a-b)$ (Plane Trig. [45]);

$$\therefore \sin\tfrac{1}{2}(A+B)=\frac{2\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

$$=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\tfrac{1}{2}C, \qquad [21]$$

$$\text{and } \sin\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\cos\tfrac{1}{2}C. \qquad [22]$$

The last four formulas are called the Gauss Formulas.

From [21] and [19] we obtain

$$\tan \tfrac{1}{2}(\mathrm{A}+\mathrm{B})=\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \operatorname{ctn} \tfrac{1}{2} \mathrm{C}; \qquad [23]$$

from [22] and [20]

$$\tan \tfrac{1}{2}(\mathrm{A}-\mathrm{B})=\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \operatorname{ctn} \tfrac{1}{2} \mathrm{C}; \qquad [24]$$

from [20] and [19]

$$\tan \tfrac{1}{2}(a+b)=\frac{\cos \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2} c; \qquad [25]$$

from [22] and [21]

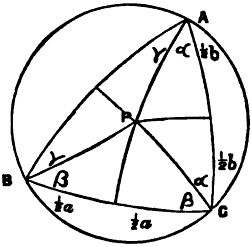
$$\tan \tfrac{1}{2}(a-b)=\frac{\sin \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\sin \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2} c. \qquad [26]$$

The last four formulas are called Napier's Analogies.

§ **23**. *To find the polar radius of a circle circumscribed about a triangle in terms of the angles.*

The centre is at the intersection of the perpendiculars to the sides at their middle points; joining the centre with the vertices we have the right triangles indicated in Fig. 4, the hypothenuses of which are each equal to the required radius (R). The triangles PAB, PBC, and PCA are isosceles, therefore the two angles marked α are equal to each other, and likewise the two angles marked β, and the two angles marked γ.

FIG. 4.



$$\text{Now } \beta=\mathrm{B}-\gamma=\mathrm{B}-(\mathrm{A}-\alpha)=\mathrm{B}-\mathrm{A}+\alpha=\mathrm{B}-\mathrm{A}+\mathrm{C}-\beta;$$

$$\therefore \beta=\tfrac{1}{2}(\mathrm{B}-\mathrm{A}+\mathrm{C})=\mathrm{S}-\mathrm{A}.$$

$$\text{Now by [3] } \tan \text{CP} = \tan \text{R} = \frac{\tan \frac{1}{2} a}{\cos \beta} = \frac{\tan \frac{1}{2} a}{\cos (\text{S} - \text{A})}$$

$$= \text{by [18]} \sqrt{\frac{-\cos \text{S} \cos (\text{S} - \text{A})}{\cos (\text{S} - \text{B}) \cos (\text{S} - \text{C})}} \times \frac{1}{\cos (\text{S} - \text{A})};$$

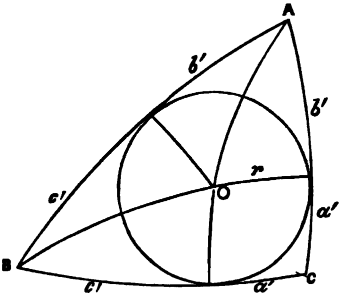
$$\therefore \tan \text{R} = \sqrt{\frac{-\cos \text{S}}{\cos (\text{S} - \text{A}) \cos (\text{S} - \text{B}) \cos (\text{S} - \text{C})}}. \quad [27]$$

Corollary.

$$\tan \tfrac{1}{2} a = \tan \text{R} \cos (\text{S} - \text{A}). \quad [28]$$

§ **24.** *To find the polar radius of the circle inscribed in a triangle in terms of the sides.*

FIG. 5.



The centre of the circle is at the intersection (O, Fig. 5) of the lines which bisect the angles, therefore the perpendiculars let fall from the centre O upon the sides a, b, and c are each equal to the required radius (r). The right triangles on each side of OA are symmetrical, and also those on each side of OB and OC, therefore the arcs marked b', b' are equal, and also those marked a', a', and c', c'.

$$\text{Now } b' = b - a' = b - (a - c') = b - (a - (c - b'))$$

$$= b - a + c - b'; \; \therefore b' = \tfrac{1}{2}(b - a + c) = s - a.$$

$$\text{Now by [4] } \tan r = \sin b' \tan \tfrac{1}{2} \text{A} = \sin (s - a) \tan \tfrac{1}{2} \text{A}$$

$$= \text{by [15] } \sin (s - a) \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}};$$

$$\therefore\ \tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}. \qquad [29]$$

Corollary.

$$\tan \tfrac{1}{2} \mathrm{A} = \frac{\tan r}{\sin (s-a)}. \qquad [30]$$

§ 25. *Solution of Oblique Triangles.*

The number of cases which can arise is equal to the number of combinations which can be formed out of the six parts, taking three at a time, or to $\frac{6 \times 5 \times 4}{\underline{|3}} = 20$, and they are the following:

(1) $a\ b\ c$, (2) $a\ b$ A, (3) $a\ b$ B, (4) $a\ b$ C, (5) $a\ c$ A,
(6) $a\ c$ B, (7) $a\ c$ C, (8) $b\ c$ A, (9) $b\ c$ B, (10) $b\ c$ C,
(11) a A B, (12) a A C, (13) a B C, (14) b A B, (15) b A C,
(16) b B C, (17) c A B, (18) c A C, (19) c B C, (20) A B C

We see, however, that the combinations (2), (3), (5), (7), and (10) form only one *distinct* case, since in each two sides and an angle opposite one of them are given; similarly (4), (6), and (8) form only one *distinct* case; (11), (12), (16), (18), and (19) form only one *distinct* case; and (13), (15), and (17) form only one *distinct* case. We have then in all only six *distinct* cases, as follows:

Case I. Given the three sides.

Case II. Given the three angles.

Case III. Given two sides and an angle opposite one of them.

Case IV. Given two angles and a side opposite one of them.

Case V. Given two sides and the included angle.

Case VI. Given one side and the two adjacent angles.

§ **26.** Cases I. and II. can be solved by [15] and [18] respectively, or I. by [30] and [29], and II. by [28] and [27]. [10] may be used as a check formula.

Example 1 (Case I.).

$$\textit{Given}\begin{cases} a = 25^\circ\ 13' \\ b = 37^\circ\ 14' \\ c = 58^\circ\ 32' \end{cases}$$

We will solve this by [30] and [29].

$$\begin{array}{rlrl}
2s = & 120^\circ\ 59' & & \\
s = & 60^\circ\ 29'.5 & \operatorname{colog}\sin s = & 0.0603 \\
s-a = & 35^\circ\ 16'.5 & \log\sin(s-a) = & 9.7615 \\
s-b = & 23^\circ\ 15'.5 & \log\sin(s-b) = & 9.5964 \\
s-c = & 1^\circ\ 57'.5 & \log\sin(s-c) = & 8.5336 \\
\hline
2s = & 120^\circ\ 59' & \log\tan^2 r = & 7.9518 \\
 & & \log\tan r = & 8.9759
\end{array}$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}A &= 9.2144 \\
\log\tan\tfrac{1}{2}B &= 9.3795 \\
\log\tan\tfrac{1}{2}C &= 0.4423
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 9^\circ\ 18' \\
\tfrac{1}{2}B &= 13^\circ\ 28'.5 \\
\tfrac{1}{2}C &= 70^\circ\ 08'.5
\end{aligned}
\qquad
\textit{Ans.}\begin{cases} A = 18^\circ\ 36' \\ B = 26^\circ\ 57' \\ C = 140^\circ\ 17' \end{cases}$$

Check.

$$\begin{array}{rrrrrr}
\log\sin A = & 9.5037 & \log\sin B = & 9.6563 & \log\sin C = & 9.8055 \\
\log\sin a = & \underline{9.6294} & \log\sin b = & \underline{9.7818} & \log\sin c = & \underline{9.9310} \\
 & 9.8743 & & 9.8745 & & 9.8745
\end{array}$$

Example 2 (Case II.).

Given $A = 102^\circ\ 14'$, $B = 54^\circ\ 32'$, $C = 89^\circ\ 06'$.

Ans. $a = 104^\circ\ 26'$, $b = 53^\circ\ 50'$, $c = 97^\circ\ 44'$.

§ **27.** CASE III. Given a, b, and A.

We find B by [10], i. e. by

$$\sin B = \frac{\sin A \sin b}{\sin a};$$

and $\frac{1}{2}c$, $\frac{1}{2}C$ by [25] and [23],

$$\text{i e. by } \tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b),$$

$$\text{and } \operatorname{ctn} \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A+B);$$

$\frac{1}{2}c$ and $\frac{1}{2}C$ being each less than 90° $\tan \frac{1}{2}c$ and $\operatorname{ctn} \frac{1}{2}C$ must be positive; therefore when each of the two supplementary values of B obtained by [10] makes $\tan \frac{1}{2}c$ and $\operatorname{ctn} \frac{1}{2}c$ positive there will be two solutions.

When $\sin A \sin b > \sin a$ there will be no solution, for in this case we shall have $\sin B > 1$.

There are many ways of determining by an inspection of the data (a, b, and A) whether there are two solutions or but one; the following, found in Chauvenet's Trigonometry, p. 197, seems to be the best:

1st. When b differs more from 90° than a, B must be in the same quadrant as b (§ 2 n), and there can be but one solution. It remains to show,

2d. That when a differs more from 90° than b, there will necessarily be two solutions. We have, by the first of [11],

$$\sin c = \frac{\cos a - \cos b \cos c}{\sin b \cos A}.$$

Two solutions exist so long as both values of c are positive, and less than 180°, that is, so long as $\sin c$ is positive. Now when a differs more from 90° than b, we have (neglecting the signs for a moment),

$$\cos a > \cos b > \cos b \cos c,$$

therefore the numerator of the above value of sin c has the sign of cos a. But by § 2 n, a and A are in the same quadrant, and cos a and cos A have the same sign ; consequently also, the numerator and denominator have the same sign, and the value of the fraction, or of sin c, is positive, as was to be proved.

Hence, *there is but one solution when the side opposite the given angle differs less from* 90° *than the other given side, and two solutions when the side opposite the given angle differs more from* 90° *than the other given side.*

NOTE. We have proved above that when the property stated in § 2 n, does not determine the quadrant of B then there are always two solutions.

When B is nearly 90° it cannot be obtained with accuracy from its sine, but we may proceed as follows:

From b and A in the right triangle ACD (Fig. 3) we can find p (v. Case V. of right triangles), and then from p and a in the right triangle BCD we can find B (v. Case II. of right triangles).

[10] may be used as a check formula.

Example. *Given* $b = 70° 21'$, $a = 51° 41'$, $A = 52° 30'$.

$b + a = 122° 02'$
$b - a = 18° 40'$
$\log \sin b = 9.9739$
$\log \sin A = 9.8995$
$\text{colog} \sin a = \underline{0.1054}$
$\log \sin B = 9.9788$
$^{*}B_1 = 72° 15'$
$B_2 = 107° 45'$

$\frac{1}{2}(b + a) = 61° 01'$
$\frac{1}{2}(b - a) = 9° 20'$
$B_1 + A = 124° 45'$
$B_2 + A = 160° 15'$
$B_1 - A = 19° 45'$
$B_2 - A = 55° 15'$
$\frac{1}{2}(B_1 + A) = 62° 22'.5$
$\frac{1}{2}(B_2 + A) = 80° 07'.5$
$\frac{1}{2}(B_1 - A) = 9° 52'.5$
$\frac{1}{2}(B_2 - A) = 27° 37'.5$

* a differs more from 90° than b ; there are therefore two solutions.

$$\begin{array}{rlrl}
\log\cos\tfrac{1}{2}(b+a) &= 9.6854 & \log\cos\tfrac{1}{2}(b+a) &= 9.6854\\
\log\tan\tfrac{1}{2}(B_1+A) &= 0.2812 & \log\tan\tfrac{1}{2}(B_2+A) &= 0.7593\\
\operatorname{colog}\cos\tfrac{1}{2}(b-a) &= \underline{0.0058} & \operatorname{colog}\cos\tfrac{1}{2}(b-a) &= \underline{0.0058}\\
\log\operatorname{ctn}\tfrac{1}{2}C_1 &= 9.9724 & \log\operatorname{ctn}\tfrac{1}{2}C_2 &= 0.4505\\
\tfrac{1}{2}C_1 &= 46^\circ\ 49' & \tfrac{1}{2}C_2 &= 19^\circ\ 31'\\
C_1 &= 93^\circ\ 38' & C_2 &= 39^\circ\ 02'\\
\log\cos\tfrac{1}{2}(B_1+A) &= 9.6662 & \log\cos\tfrac{1}{2}(B_2+A) &= 9.2342\\
\log\tan\tfrac{1}{2}(b+a) &= 0.2565 & \log\tan\tfrac{1}{2}(b+a) &= 0.2565\\
\operatorname{colog}\cos\tfrac{1}{2}(B_1-A) &= \underline{0.0064} & \operatorname{colog}\cos\tfrac{1}{2}(B_2-A) &= 0.0526\\
\log\tan\tfrac{1}{2}c_1 &= 9.9291 & \log\tan\tfrac{1}{2}c_2 &= 9.5433\\
\tfrac{1}{2}c_1 &= 40^\circ\ 21' & \tfrac{1}{2}c_2 &= 19^\circ 15'.5\\
c_1 &= 80^\circ\ 42' & c_2 &= 38^\circ\ 31'
\end{array}$$

Check.

$$\begin{array}{rlrlrl}
\log\sin B &= 9.9788 & \log\sin C_1 &= 9.9991 & \log\sin C_2 &= 9.7992\\
\log\sin b &= \underline{9.9739} & \log\sin c_1 &= \underline{9.9942} & \log\sin c_2 &= \underline{9.7943}\\
 & 0.0049 & & 0.0049 & & 0.0049
\end{array}$$

§ **28.** Case IV. Given A, B, and a.

We find b by [10], i. e. by

$$\sin b = \frac{\sin B \sin a}{\sin A};$$ when b is near 90° it may be found in a manner exactly similar to that by which B when near 90° was found in Case III. c and C may be found as in Case III.

When each of the two values of b found by either of the above formulas makes $\tan\frac{1}{2}c$ and $\operatorname{ctn}\frac{1}{2}C$ positive then there will be two solutions.

When $\dfrac{\sin B \sin a}{\sin A} > 1$ there is no solution.

The supplements of the parts A, B, and a are respectively the parts a', b', and A' of the *polar triangle ;* the polar triangle then comes under the head of Case III., and the rule given in the last section tells us whether it has one or two solutions. Our triangle evidently has the same number of solutions as its polar.

Example. Given $A = 159^\circ 43'$, $B = 123^\circ 40'$, $a = 159^\circ 50'$.

$$\textit{Ans. } b = \begin{matrix} 124^\circ\ 08' \\ 55^\circ\ 52' \end{matrix}, \quad c = \begin{matrix} 137^\circ\ 20' \\ 65^\circ\ 39' \end{matrix}, \quad C = \begin{matrix} 137^\circ\ 04' \\ 113^\circ\ 39' \end{matrix}.$$

§ **29.** CASE V. Given a, b, and C.

By the following formulas ([23] and [24])

$$\tan \tfrac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \operatorname{ctn} \tfrac{1}{2} C,$$

$$\tan \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \operatorname{ctn} \tfrac{1}{2} C,$$

we may find $\frac{1}{2} (A + B)$ and $\frac{1}{2} (A - B)$, and therefore A and B ; we can now find c either by [25] or [26], i. e.

$$\text{by } \tan \tfrac{1}{2} (a + b) = \frac{\cos \frac{1}{2} (A - B)}{\cos \frac{1}{2} (A + B)} \tan \tfrac{1}{2} c,$$

$$\text{or } \tan \tfrac{1}{2} (a - b) = \frac{\sin \frac{1}{2} (A - B)}{\sin \frac{1}{2} (A + B)} \tan \tfrac{1}{2} c.$$

[10] may be used as a check formula.

Example. Given $a = 56^\circ 20'$, $b = 20^\circ 17'$, $C = 114^\circ 20'$.

$a + b = 76^\circ\ 37'$ $\qquad \frac{1}{2} (a + b) = 38^\circ\ 18'.5$

$a - b = 36^\circ\ 23'$ $\qquad \frac{1}{2} (a - b) = 18^\circ\ 01'.5$

$\frac{1}{2} C = 57^\circ\ 10'$

$$\log \cos \tfrac{1}{2}(a-b) = 9.9781$$
$$\log \text{ctn} \tfrac{1}{2} C = 9.8097$$
$$\text{colog} \cos \tfrac{1}{2}(a+b) = \underline{0.1053}$$
$$\log \tan \tfrac{1}{2}(A+B) = 9.8931$$
$$\tfrac{1}{2}(A+B) = 38^\circ\ 01'$$
$$A = 55^\circ\ 52'$$
$$\log \tan \tfrac{1}{2}(a+b) = 9.8976$$
$$\log \cos \tfrac{1}{2}(A+B) = 9.8964$$
$$\text{colog} \cos \tfrac{1}{2}(A-B) = \underline{0.0214}$$
$$\log \tan \tfrac{1}{2} c = 9.8154$$
$$\tfrac{1}{2} c = 33^\circ\ 10'$$
$$c = 66^\circ\ 20'$$

$$\log \sin \tfrac{1}{2}(a-b) = 9.4906$$
$$\log \text{ctn} \tfrac{1}{2} C = 9.8097$$
$$\text{colog} \sin \tfrac{1}{2}(a+b) = \underline{0.2077}$$
$$\log \tan \tfrac{1}{2}(A-B) = 9.5080$$
$$\tfrac{1}{2}(A-B) = 17^\circ\ 51'$$
$$B = 20^\circ\ 10'$$

Check.

$$\log \sin A = 9.9179$$
$$\log \sin a = \underline{9.9203}$$
$$9.9976$$
$$\log \sin B = 9.5375$$
$$\log \sin b = \underline{9.5399}$$
$$9.9976$$
$$\log \sin C = 9.9596$$
$$\log \sin c = \underline{9.9618}$$
$$9.9978$$

§ **30.** CASE VI. Given A, B, and c.

The formulas used in the solution of Case V. can also be used here; [25] and [26] may be used to find a and b, and either [23] or [24] to find C.

EXAMPLES.

	Given			*Compute*		
	a	b	c	A	B	C
1.	124° 12′.5	54° 18′	97° 12′.5	127° 22′	51° 18′	72° 27′
2.	93° 45′	27° 16′	88° 12′	101° 48′	26° 42′	78° 42′
3.	82° 34′	27° 16′	89° 12′	75° 12′	26° 32′	102° 51′
4.	82° 11′	64° 19′.5	31° 31′.5	119° 44′	52° 16′	27° 16′

	Given			*Compute*		
	A	B	C	*a*	*b*	*c*
5.	20° 10′	55° 52′	114° 20′	20° 16′	56° 18′	66° 18′
6.	130° 00′	110° 00′	80° 00′	139° 21′	126° 58′	56° 52′
7.	55° 42′	45° 44′	135° 16′	82° 17′	59° 12′	122° 25
8.	4° 24′	8° 28′	172° 18′	31° 14′	84° 12′	115° 08
	a	*b*	A	*c*	B	C
9.	69° 50′	46° 42′	32° 54′	109° 39′	24° 55′	146° 58′
10.	40° 20′	70° 40′	40° 00′	81° 54′ 48° 54′	69° 34′ 110° 26′	100° 32′ 48° 26′
11.	50° 00′	40° 00′	80° 00′	42° 10′	55° 43′	59° 38′
12.	150° 57′	134° 16′	144° 23′	23° 57′ 55° 42′	59° 11′ 120° 49′	29° 09′ 97° 44′
	A	B	*a*	*b*	*c*	C
13.	110° 10′	133° 18′	147° 06′	155° 05′	33° 02′	70° 21′
14.	113° 39′	123° 40′	65° 40′	124° 07′	159° 50′	159° 44′
15.	159° 43′	123° 40′	159° 50′	55 52′ 124 08′	137° 22′ 65° 39′	137° 04′ 113° 39′
16.	100° 02′	98° 30′	95° 21′	90°	147° 38′	147° 58′
	a	*b*	C	*c*	A	B
17.	35° 37′	59° 12′	124° 18′	82° 16′	29° 02′	45° 44′
18.	124° 07′	88° 12′	50° 02′	59° 04′	132° 18′	63° 15′
19.	88° 12′	120° 56′	47° 42′	55° 56′	63° 15′	129° 59′
20.	47° 42′	63° 15′	59° 04′	50° 00′	55° 53′	88° 13′
	A	B	*c*	*a*	*b*	C
21.	57° 35′	120° 48′	124° 18′	44° 44′.5	134° 15′.5	97° 42′
22.	39° 45′	26° 59′	154° 47′	121° 27′.5	37° 15′.5	161° 22′
23.	107° 33′	128° 42′	124° 13′	82° 47′	125° 42′	127° 22′
24.	82° 48′	125° 42′	52° 38′	107° 34′	128° 42′	55° 48′

25. Find the length in geographical miles of the great circle arc joining Sandy Hook (New York Harbor), lat. 40° 28′, long. 74° 08′, with Cork Harbor (Queenstown), lat. 51° 47′, long. 8° 11′. *Ans.* 2726 m.

Remark. The lengths of the routes travelled by steamers of the White Star Line are as follows:

Queenstown to Sandy Hook.

Route A (summer), 2783 miles.
Route B (winter), 2855 miles.

Sandy Hook to Queenstown.

Route C, . . . 2889 miles.

26. Find the distance from Sandy Hook (*v.* Ex. 25) to the Giants' Causeway (north coast of Ireland), lat. 55° 18′, long. 6° 24′. *Ans.* 2745 m.

APPENDIX I.

DEFINITIONS AND EXPLANATIONS OF SOME OF THE TERMS USED IN SPHERICAL ASTRONOMY.

§ **1.** All heavenly bodies may be considered to be projected upon the concave surface of a sphere of indefinite radius, the eye of the observer being at the centre of the sphere. This sphere is called the *Celestial Sphere.*

§ **2.** The axis of the earth, or the straight line which joins its poles, is also the axis of the celestial sphere, and the points in which it cuts the latter are called the *poles* of the celestial sphere.

§ **3.** The great circle in which the plane of the earth's equator cuts the celestial sphere is called the *celestial equator*, or the *equinoctial.*

§ **4.** Great circles which pass through the poles of the celestial sphere are called *hour circles*, or *circles of declination;* they are evidently perpendicular to the equinoctial.

§ **5.** The *celestial meridian* of a place on the earth's surface is the circle in which the plane of the terrestial meridian of the place cuts the celestial sphere.

§ **6.** Since the earth makes a complete rotation about its axis once in twenty-four hours, every point of the celestial meridian will move through a complete circumference, or arc of 360°, in the same time; hence angles are sometimes measured in time, thus 24 h. = 360°, 1 h. = 15°.

§ **7.** The *hour angle* of a heavenly body is the inclination of the hour circle (circle of declination) which passes through the body to the celestial meridian, and is measured by the arc of the celestial equator (equinoctial) included between these two circles; hour angles are measured positive from the celestial meridian towards the west, from 0° to 360°, or 0 h. to 24 h. The hour angle of the sun is the solar time. When, for example, the hour angle of the sun is 30° or 2 h., the solar time is 2 o'c. P. M.; and when the hour angle is -30° the time is 10 o'c. A. M.

§ **8.** The *sensible horizon* of an observer is the plane through his feet and tangent to the earth's surface; and his *rational horizon* is a plane parallel to the first through the centre of the earth. These two planes intersect the surface of the celestial sphere in two circles; but the radius of the celestial sphere is so immense in comparison with the radius of the earth that the two circles will sensibly coincide and form one great circle called the *celestial horizon*. The earth then, when compared with the celestial sphere, is to be regarded as only a point at its centre.

§ **9.** The straight line through the feet of the observer and perpendicular to the plane of his horizon is called a vertical line, and the upper and lower points in which it cuts the celestial sphere are called the *zenith* and *nadir* respectively.

§ **10.** *Vertical circles* are great circles which pass through the zenith and nadir. The celestial meridian is evidently a vertical circle, and passes through the north and south points of the horizon.

The *prime vertical* is that vertical circle which is perpendicular to the plane of the celestial meridian; it therefore cuts the celestial horizon in its east and west points.

§ **11.** The great circle of the celestial sphere in which the sun appears to move, or in which to an observer on the sun the earth would appear to move, is called the *ecliptic.* The points in which a straight line through the earth and perpendicular to the plane of the ecliptic cuts the celestial sphere are called the *poles* of the ecliptic; great circles through these points are called *circles of latitude.*

§ **12.** The *vernal* and *autumnal equinoxes* are the points in which the ecliptic cuts the equinoctial.

§ **13.** The inclination of the ecliptic to the equinoctial is called the *obliquity of the ecliptic,* and is denoted by the letter e; it is equal to 23° 27′.

§ **14.** Spherical Coördinates.

The position of any point on the surface of a sphere is determined when its angular distances from any two fixed great circles of the sphere are known. Thus the position of any point on the earth's surface is known when we have determined its latitude and longitude, i. e. its angular distances from the equator and the prime meridian.

§ **15.** For the purpose of denoting the position of a point on the celestial sphere astronomers make use of three sets of coördinate circles, or circles of reference, as follows:

I. *The celestial horizon, and the celestial meridian of the place.*

The angle made by the vertical circle through a body with the celestial meridian is called the *azimuth;* this is generally reckoned from the south point of the horizon to the right hand, from 0° to 360°.

The angular distance of the body above the celestial horizon measured on the vertical circle through the body is called the *altitude.*

The distance of a body from the zenith measured on the vertical circle is called the zenith distance; it is evidently the complement of the altitude.

II. *The equinoctial, and the hour circle* (circle of declination) *through the vernal equinox.*

The angle made by the hour circle through a body with the hour circle through the vernal equinox is called the *right ascension;* this is always reckoned from the vernal equinox towards the east, from 0° to 360° or 0 h. to 24 h.

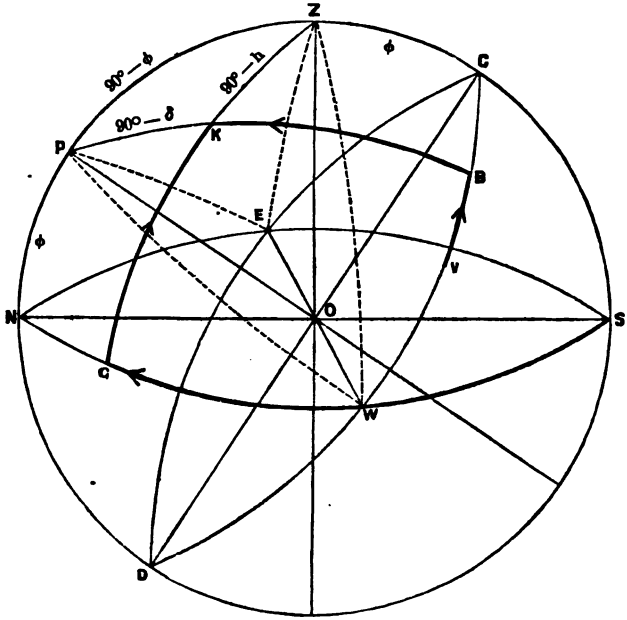
The angular distance of a body from the plane of the equinoctial measured on the hour circle of the body is called the *declination;* it is reckoned to the north and south, and cannot therefore exceed 90°. The distance of a body from either pole measured on its hour circle is called the *polar distance;* it is evidently the complement of the declination.

III. *The ecliptic, and the circle of latitude through the vernal equinox.*

The angle made by the circle of latitude through the body and the circle of latitude through the vernal equinox is called the *longitude;* it is reckoned from the vernal equinox towards the east, from 0° to 360°.

The angular distance of a body from the ecliptic measured on its circle of latitude is called the *latitude.*

FIG. 1.



§ **16.** The position of a body in the heavens may then be denoted either by its *altitude* and *azimuth,* or by its *right ascension* and *declination,* or by its *latitude* and *longitude.*

Care must be taken not to confound the latitude and longitude of a place on the earth's surface with the latitude and longitude of a heavenly body.

The sun being always in the ecliptic, its latitude is always 0° ; its longitude changes by 360° during the year: on March 21, it is 0° ; on June 21, it is 90° ; on September 21, it is 180° ; on December 21, it is 270° ; and on March 21, again, it is 360°, or 0°.

When a body rises or sets its altitude is evidently 0°, and when the altitude is greatest the body is on the celestial meridian and the azimuth is either 0° or 180°.

§ **17.** In Fig. 1, O represents the earth, NESW the celestial horizon of the observer, Z the zenith, DECW the equinoctial, P the elevated pole (i. e. that one which can be seen by the observer), and V the the vernal equinox.

If K denotes some celestial body, then the elements of its position are as follows:

[The order of the letters denotes the direction in which the elements are measured.]

GK = altitude (h),

ZK = zenith distance,

SWG = azimuth (a),

BK = declination (δ),

PK = polar distance,

VB = right ascension (α),

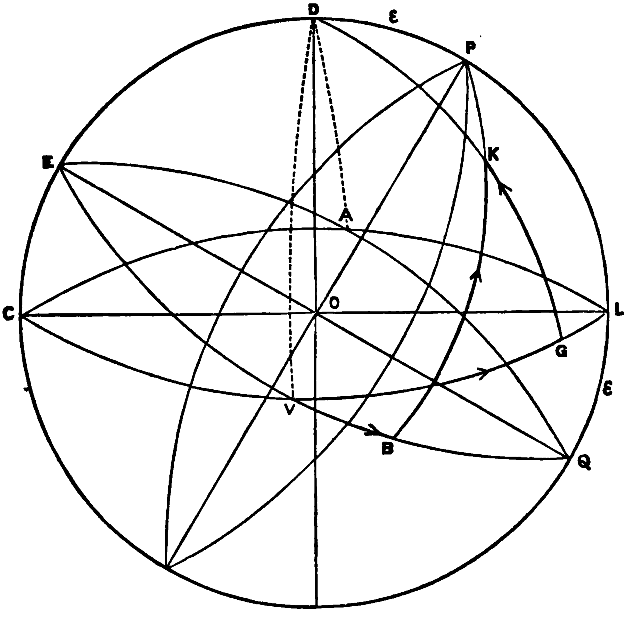
CPB (CB) = hour angle (t),

CZ = NP = latitude (ϕ) of the place of observation.

The triangle PZK is called the *astronomical triangle*, the elements of which are $90° - \phi$, $90 - h$, $90 - \delta$, t, the angle K called the position angle, and the angle KZP $= 180° - a$. The solution of this triangle from sufficient data will enable us to answer many of the questions which are likely to arise in Spherical Astronomy.

§ **18.** In Fig. 2 let CALV denote the ecliptic, EAQV the equinoctial, D and P their respective poles, and K the position of a body in the heavens.

FIG. 2.



Hence it follows that A and V denote the autumnal and vernal equinoxes, $LQ = DP$ the obliquity of the ecliptic $(e = 23^\circ\ 27')$; and the elements of the position of K are

$VB =$ the right ascension (α),
$BK =$ the declination (δ),
$VG =$ the celestial longitude,
$GK =$ the celestial latitude.

APPENDIX II.

COLLECTION OF FORMULAS AND OTHER USEFUL MATTER, FOR EASY REFERENCE.

Right Triangles.

§ 6.

$$\cos c = \cos a \cos b. \qquad [1]$$

$$\sin B = \frac{\sin b}{\sin c}; \quad \sin A = \frac{\sin a}{\sin c}. \qquad [2]$$

$$\cos B = \frac{\tan a}{\tan c}; \quad \cos A = \frac{\tan b}{\tan c}. \qquad [3]$$

$$\tan B = \frac{\tan b}{\sin a}; \quad \tan A = \frac{\tan a}{\sin b}. \qquad [4]$$

$c < 90°$ when a and b are in the same quadrant.

$c > 90°$ " " " " " different quadrants.

An oblique angle and its opposite side are in the same quadrant.

§ 7.

$$\cos b = \frac{\cos B}{\sin A}; \quad \cos a = \frac{\cos A}{\sin B}. \qquad [5]$$

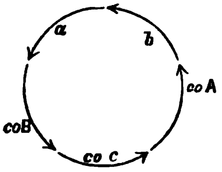
$$\cos c = \text{ctn } A \text{ ctn } B. \qquad [6]$$

Napier's Rules.

§ 8.

(1) The *sine* of any part is equal to the product of the (*t*)*angents* of the (*a*)*djacent* parts.

(2) The *sine* of any part is equal to the product of the (*c*)*osines* of the (*o*)*pposite* parts.



§10.

Given	*Sought*
Case I. a and b.	$\cos c = \cos a \cos b$, $\tan A = \dfrac{\tan a}{\sin b}$, $\tan B = \dfrac{\tan b}{\sin a}$.
Case II. a and c.	$\cos b = \dfrac{\cos c}{\cos a}$, $\sin A = \dfrac{\sin a}{\sin c}$, $\cos B = \dfrac{\tan a}{\tan c}$.
Case III. a and B.	$\cos A = \cos a \sin B$, $\tan b = \sin a \tan B$, $\tan c = \dfrac{\tan a}{\cos B}$.
Case IV. a and A.	$\sin B = \dfrac{\cos A}{\cos a}$, $\sin b = \dfrac{\tan a}{\tan A}$, $\sin c = \dfrac{\sin a}{\sin A}$.
Case V. c and A.	$\text{ctn}\, B = \cos c \tan A$, $\tan b = \tan c \cos A$, $\sin a = \sin c \sin A$.
Case VI. A and B.	$\cos a = \dfrac{\cos A}{\sin B}$, $\cos b = \dfrac{\cos B}{\sin A}$, $\cos c = \text{ctn}\, A \,\text{ctn}\, B$.

Check Formulas.

CASE I. $\cos c = \text{ctn}\, A \,\text{ctn}\, B$.

CASE II. $\cos b = \dfrac{\cos B}{\sin A}$.

CASE III. $\tan b = \tan c \cos A$.

CASE IV. $\sin b = \sin c \sin B$.

CASE V. $\tan B = \dfrac{\tan b}{\sin a}$.

CASE VI. $\cos c = \cos a \cos b$.

§12. *Special Formulas for Case II.* (given a and c).

$$\left.\begin{aligned}
&\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2} (a + c) \tan \tfrac{1}{2} (c - a), && (a)\\
&\tan^2 (45^\circ - \tfrac{1}{2} A) = \tan \tfrac{1}{2} (c - a) \,\text{ctn}\, \tfrac{1}{2} (c + a), && (b)\\
&\tan^2 \tfrac{1}{2} B = \sin (c - a) \div \sin (c + a). && (c)
\end{aligned}\right\} [7]$$

§ 14. In Case IV. (given a and A) two solutions are always possible, viz.:

(1) b acute, B acute, c and a alike.

(2) b obtuse, B obtuse, c and a unlike.

Special Formulas for Case IV.

$$\left.\begin{array}{ll}\tan^2(45^\circ-\tfrac{1}{2}\mathrm{B})=\tan\tfrac{1}{2}(\mathrm{A}- -a)\tan\tfrac{1}{2}(\mathrm{A}+a). & (a)\\ \tan^2(45^\circ-\tfrac{1}{2}b)=\sin(\mathrm{A}-a)\div\sin(\mathrm{A}+a). & (b)\\ \tan^2(45^\circ-\tfrac{1}{2}c)=\tan\tfrac{1}{2}(\mathrm{A}-a)\operatorname{ctn}\tfrac{1}{2}(\mathrm{A}+a). & (c)\end{array}\right\}[8]$$

§ 16. *Special Formulas for Case VI.* (given A and B).

$$\left.\begin{array}{ll}\operatorname{ctn}^2\tfrac{1}{2}a=\dfrac{\tan\tfrac{1}{2}(90^\circ+\mathrm{B}-\mathrm{A})}{\tan\tfrac{1}{2}(\mathrm{B}+\mathrm{A}-90^\circ)}. & (a)\\ \operatorname{ctn}^2\tfrac{1}{2}b=\dfrac{\tan\tfrac{1}{2}(90^\circ+\mathrm{A}-\mathrm{B})}{\tan\tfrac{1}{2}(\mathrm{B}+\mathrm{A}-90^\circ)}. & (b)\\ \operatorname{ctn}^2\tfrac{1}{2}c=\dfrac{-\cos(\mathrm{B}-\mathrm{A})}{\cos(\mathrm{B}+\mathrm{A})}. & (c)\end{array}\right\}[9]$$

The General Triangle.

§ 17. $$\frac{\sin \mathrm{A}}{\sin a}=\frac{\sin \mathrm{B}}{\sin b}=\frac{\sin \mathrm{C}}{\sin c}. \qquad [10]$$

§ 18. $$* \cos a=\cos b\cos c+\sin b\sin c\cos \mathrm{A}. \qquad [11]$$

§ 19. $$* -\cos \mathrm{A}=\cos \mathrm{B}\cos \mathrm{C}-\sin \mathrm{B}\sin \mathrm{C}\cos a. \qquad [12]$$

§ 20.

$$* \cos\tfrac{1}{2}\mathrm{A}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}. \qquad [13]$$

$$* \sin\tfrac{1}{2}\mathrm{A}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}. \qquad [14]$$

$$* \tan\tfrac{1}{2}\mathrm{A}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}. \qquad [15]$$

§ **21.**

$$^{*}\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}. \qquad [16]$$

$$^{*}\sin\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}. \qquad [17]$$

$$^{*}\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}. \qquad [18]$$

§ **22.**

Gauss Formulas.

$$\cos\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\tfrac{1}{2}C. \qquad [19]$$

$$\cos\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\tfrac{1}{2}C. \qquad [20]$$

$$\sin\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\tfrac{1}{2}C. \qquad [21]$$

$$\sin\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\cos\tfrac{1}{2}C. \qquad [22]$$

§ **22.**

Napier's Analogies.

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\operatorname{ctn}\tfrac{1}{2}C. \qquad [23]$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\operatorname{ctn}\tfrac{1}{2}C. \qquad [24]$$

$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c. \qquad [25]$$

$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c. \qquad [26]$$

§ **23.**

$$\tan R=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}. \qquad [27]$$

$$\tan\tfrac{1}{2}a=\tan R\cos(S-A). \qquad [28]$$

---

* *v.* note, p. 80, Plane Trig.

§ 24.
$$\begin{cases} \tan r = \sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. & [29] \\ \tan \tfrac{1}{2}\mathrm{A} = \dfrac{\tan r}{\sin(s-a)}. & [30] \end{cases}$$

Solution of Oblique Triangles.

[10] may be used as a check formula in each case.

CASE I. Given a, b, and c; use [29] and [30].

CASE II. Given A, B, and C; use [27] and [28].

CASE III. Given a, b, and A; use [10], [25], and [23].

Only *one* solution when the side opposite the given angle differs less from 90° than the other given side, and *two* solutions when the side opposite the given angle differs more from 90° than the other given side.

CASE IV. Given A, B, and a; use [10], [25], and [23].

There will be one or two solutions according as the polar triangle (*v.* Case III.) has one or two solutions.

CASE V. Given a, b, and C; use [23], [24], and [25] or [26].

CASE VI. Given A, B, and c; use [25], [26], and [23] or [24].

MATHEMATICAL TABLES

CHIEFLY TO FOUR FIGURES

FIRST SERIES

BY

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Logarithms.

N	0	1	2	3	4	5	6	7	8	9	P. P.				
											1	2	3	4	5
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	5	7	9
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	6
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4

Logarithms.

N	0	1	2	3	4	5	6	7	8	9	P. P. 1·2·3·4·5
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1·2·2·3·4
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1·2·2·3·4
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1·2·2·3·4
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1·1·2·3·4
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1·1·2·3 4
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1·1·2·3·4
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1·1·2·3·4
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1·1·2·3·3
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1·1·2·3·3
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1·1·2·3·3
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1·1·2·3·3
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1·1·2·3·3
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1·1·2·3·3
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1·1·2·3·3
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1·1·2·3·3
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1·1·2·2·3
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1·1·2·2·3
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1·1·2·2·3
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1·1·2·2·3
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1·1·2·2·3
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1·1·2·2·3
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1·1·2·2·3
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1·1·2·2·3
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1·1·2·2·3
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1·1·2·2·3
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1·1·2·2·3
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1·1·2·2·3
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1·1·2·2·3
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1·1·2·2·3
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1·1·2·2·3
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1·1·2·2·3
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1·1·2·2·3
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0·1·1·2·2
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0·1·1·2·2
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0·1·1·2·2
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0·1·1·2·2
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0·1·1·2·2
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0·1·1·2·2
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0·1·1·2·2
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1·1·2·2
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0·1·1·2·2
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0·1·1·2·2
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0·1·1·2·2
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0·1·1·2·2
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0·1·1·2·2

Logarithms.

N	0	1	2	3	4	5	6	7	8	9	10
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	0043
101	0043	0048	0052	0056	0060	0065	0069	0073	0077	0082	0086
102	0086	0090	0095	0099	0103	0107	0111	0116	0120	0124	0128
103	0128	0133	0137	0141	0145	0149	0154	0158	0162	0166	0170
104	0170	0175	0179	0183	0187	0191	0195	0199	0204	0208	0212
105	0212	0216	0220	0224	0228	0233	0237	0241	0245	0249	0253
106	0253	0257	0261	0265	0269	0273	0278	0282	0286	0290	0294
107	0294	0298	0302	0306	0310	0314	0318	0322	0326	0330	0334
108	0334	0338	0342	0346	0350	0354	0358	0362	0366	0370	0374
109	0374	0378	0382	0386	0390	0394	0398	0402	0406	0410	0414
110	0414	0418	0422	0426	0430	0434	0438	0441	0445	0449	0453
111	0453	0457	0461	0465	0469	0473	0477	0481	0484	0488	0492
112	0492	0496	0500	0504	0508	0512	0515	0519	0523	0527	0531
113	0531	0535	0538	0542	0546	0550	0554	0558	0561	0565	0569
114	0569	0573	0577	0580	0584	0588	0592	0596	0599	0603	0607
115	0607	0611	0615	0618	0622	0626	0630	0633	0637	0641	0645
116	0645	0648	0652	0656	0660	0663	0667	0671	0674	0678	0682
117	0682	0686	0689	0693	0697	0700	0704	0708	0711	0715	0719
118	0719	0722	0726	0730	0734	0737	0741	0745	0748	0752	0755
119	0755	0759	0763	0766	0770	0774	0777	0781	0785	0788	0792
120	0792	0795	0799	0803	0806	0810	0813	0817	0821	0824	0828
121	0828	0831	0835	0839	0842	0846	0849	0853	0856	0860	0864
122	0864	0867	0871	0874	0878	0881	0885	0888	0892	0896	0899
123	0899	0903	0906	0910	0913	0917	0920	0924	0927	0931	0934
124	0934	0938	0941	0945	0948	0952	0955	0959	0962	0966	0969
125	0969	0973	0976	0980	0983	0986	0990	0993	0997	1000	1004
126	1004	1007	1011	1014	1017	1021	1024	1028	1031	1035	1038
127	1038	1041	1045	1048	1052	1055	1059	1062	1065	1069	1072
128	1072	1075	1079	1082	1086	1089	1092	1096	1099	1103	1106
129	1106	1109	1113	1116	1119	1123	1126	1129	1133	1136	1139
130	1139	1143	1146	1149	1153	1156	1159	1163	1166	1169	1173
131	1173	1176	1179	1183	1186	1189	1193	1196	1199	1202	1206
132	1206	1209	1212	1216	1219	1222	1225	1229	1232	1235	1239
133	1239	1242	1245	1248	1252	1255	1258	1261	1265	1268	1271
134	1271	1274	1278	1281	1284	1287	1290	1294	1297	1300	1303
135	1303	1307	1310	1313	1316	1319	1323	1326	1329	1332	1335
136	1335	1339	1342	1345	1348	1351	1355	1358	1361	1364	1367
137	1367	1370	1374	1377	1380	1383	1386	1389	1392	1396	1399
138	1399	1402	1405	1408	1411	1414	1418	1421	1424	1427	1430
139	1430	1433	1436	1440	1443	1446	1449	1452	1455	1458	1461
140	1461	1464	1467	1471	1474	1477	1480	1483	1486	1489	1492
141	1492	1495	1498	1501	1504	1508	1511	1514	1517	1520	1523
142	1523	1526	1529	1532	1535	1538	1541	1544	1547	1550	1553
143	1553	1556	1559	1562	1565	1569	1572	1575	1578	1581	1584
144	1584	1587	1590	1593	1596	1599	1602	1605	1608	1611	1614
145	1614	1617	1620	1623	1626	1629	1632	1635	1638	1641	1644
146	1644	1647	1649	1652	1655	1658	1661	1664	1667	1670	1673
147	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700	1703
148	1703	1706	1708	1711	1714	1717	1720	1723	1726	1729	1732
149	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758	1761

Logarithms.

N	0	1	2	3	4	5	6	7	8	9	10
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787	1790
151	1790	1793	1796	1798	1801	1804	1807	1810	1813	1816	1818
152	1818	1821	1824	1827	1830	1833	1836	1838	1841	1844	1847
153	1847	1850	1853	1855	1858	1861	1864	1867	1870	1872	1875
154	1875	1878	1881	1884	1886	1889	1892	1895	1898	1901	1903
155	1903	1906	1909	1912	1915	1917	1920	1923	1926	1928	1931
156	1931	1934	1937	1940	1942	1945	1948	1951	1953	1956	1959
157	1959	1962	1965	1967	1970	1973	1976	1978	1981	1984	1987
158	1987	1989	1992	1995	1998	2000	2003	2006	2009	2011	2014
159	2014	2017	2019	2022	2025	2028	2030	2033	2036	2038	2041
160	2041	2044	2047	2049	2052	2055	2057	2060	2063	2066	2068
161	2068	2071	2074	2076	2079	2082	2084	2087	2090	2092	2095
162	2095	2098	2101	2103	2106	2109	2111	2114	2117	2119	2122
163	2122	2125	2127	2130	2133	2135	2138	2140	2143	2146	2148
164	2148	2151	2154	2156	2159	2162	2164	2167	2170	2172	2175
165	2175	2177	2180	2183	2185	2188	2191	2193	2196	2198	2201
166	2201	2204	2206	2209	2212	2214	2217	2219	2222	2225	2227
167	2227	2230	2232	2235	2238	2240	2243	2245	2248	2251	2253
168	2253	2256	2258	2261	2263	2266	2269	2271	2274	2276	2279
169	2279	2281	2284	2287	2289	2292	2294	2297	2299	2302	2304
170	2304	2307	2310	2312	2315	2317	2320	2322	2325	2327	2330
171	2330	2333	2335	2338	2340	2343	2345	2348	2350	2353	2355
172	2355	2358	2360	2363	2365	2368	2370	2373	2375	2378	2380
173	2380	2383	2385	2388	2390	2393	2395	2398	2400	2403	2405
174	2405	2408	2410	2413	2415	2418	2420	2423	2425	2428	2430
175	2430	2433	2435	2438	2440	2443	2445	2448	2450	2453	2455
176	2455	2458	2460	2463	2465	2467	2470	2472	2475	2477	2480
177	2480	2482	2485	2487	2490	2492	2494	2497	2499	2502	2504
178	2504	2507	2509	2512	2514	2516	2519	2521	2524	2526	2529
179	2529	2531	2533	2536	2538	2541	2543	2545	2548	2550	2553
180	2553	2555	2558	2560	2562	2565	2567	2570	2572	2574	2577
181	2577	2579	2582	2584	2586	2589	2591	2594	2596	2598	2601
182	2601	2603	2605	2608	2610	2613	2615	2617	2620	2622	2625
183	2625	2627	2629	2632	2634	2636	2639	2641	2643	2646	2648
184	2648	2651	2653	2655	2658	2660	2662	2665	2667	2669	2672
185	2672	2674	2676	2679	2681	2683	2686	2688	2690	2693	2695
186	2695	2697	2700	2702	2704	2707	2709	2711	2714	2716	2718
187	2718	2721	2723	2725	2728	2730	2732	2735	2737	2739	2742
188	2742	2744	2746	2749	2751	2753	2755	2758	2760	2762	2765
189	2765	2767	2769	2772	2774	2776	2778	2781	2783	2785	2788
190	2788	2790	2792	2794	2797	2799	2801	2804	2806	2808	2810
191	2810	2813	2815	2817	2819	2822	2824	2826	2828	2831	2833
192	2833	2835	2838	2840	2842	2844	2847	2849	2851	2853	2856
193	2856	2858	2860	2862	2865	2867	2869	2871	2874	2876	2878
194	2878	2880	2882	2885	2887	2889	2891	2894	2896	2898	2900
195	2900	2903	2905	2907	2909	2911	2914	2916	2918	2920	2923
196	2923	2925	2927	2929	2931	2934	2936	2938	2940	2942	2945
197	2945	2947	2949	2951	2953	2956	2958	2960	2962	2964	2967
198	2967	2969	2971	2973	2975	2978	2980	2982	2984	2986	2989
199	2989	2991	2993	2995	2997	2999	3002	3004	3006	3008	3010

Logarithms of Sums and Differences.

A	6.	7.	8.	9.	0.	1.	2.	3.
00	0.0000	0.0004	0.0043	0.0414 9	0.3010 50	1.0414 91	2.0043	3.0004
01	0.0000	0.0004	0.0044	0.0423 9	0.3061 51	1.0505 91	2.0142	3.0104
02	0.0000	0.0005	0.0045	0.0432 9	0.3111 51	1.0596 91	2.0241	3.0204
03	0.0000	0.0005	0.0046	0.0442 10	0.3163 52	1.0687 91	2.0340	3.0304
04	0.0000	0.0005	0.0047	0.0452 10	0.3215 52	1.0779 92	2.0439	3.0404
05	0.0000	0.0005	0.0048	0.0462 10	0.3267 53	1.0871 92	2.0539	3.0504
06	0.0000	0.0005	0.0050	0.0472 10	0.3321 53	1.0963 92	2.0638	3.0604
07	0.0001	0.0005	0.0051	0.0482 11	0.3374 54	1.1055 92	2.0737	3.0704
08	0.0001	0.0005	0.0052	0.0493 11	0.3429 55	1.1147 92	2.0836	3.0804
09	0.0001	0.0005	0.0053	0.0504 11	0.3484 55	1.1239 92	2.0935	3.0904
10	0.0001	0.0005	0.0054	0.0515 11	0.3539 56	1.1332 93	2.1034	3.1003
11	0.0001	0.0006	0.0056	0.0526 11	0.3595 56	1.1425 93	2.1134	3.1103
12	0.0001	0.0006	0.0057	0.0538 12	0.3652 57	1.1518 93	2.1233	3.1203
13	0.0001	0.0006	0.0058	0.0550 12	0.3709 57	1.1611 93	2.1332	3.1303
14	0.0001	0.0006	0.0060	0.0562 12	0.3766 58	1.1704 93	2.1431	3.1403
15	0.0001	0.0006	0.0061	0.0574 12	0.3825 59	1.1797 93	2.1531	3.1503
16	0.0001	0.0006	0.0062	0.0586 13	0.3884 59	1.1891 94	2.1630	3.1603
17	0.0001	0.0006	0.0064	0.0599 13	0.3943 60	1.1984 94	2.1729	3.1703
18	0.0001	0.0007	0.0065	0.0612 13	0.4003 60	1.2078 94	2.1829	3.1803
19	0.0001	0.0007	0.0067	0.0625 13	0.4063 61	1.2172 94	2.1928	3.1903
20	0.0001	0.0007	0.0068	0.0639 14	0.4124 61	1.2266 94	2.2027	3.2003
21	0.0001	0.0007	0.0070	0.0653 14	0.4186 62	1.2360 94	2.2127	3.2103
22	0.0001	0.0007	0.0071	0.0667 14	0.4248 62	1.2454 94	2.2226	3.2203
23	0.0001	0.0007	0.0073	0.0681 15	0.4311 63	1.2548 94	2.2325	3.2303
24	0.0001	0.0008	0.0075	0.0696 15	0.4374 63	1.2643 95	2.2425	3.2402
25	0.0001	0.0008	0.0077	0.0711 15	0.4438 64	1.2738 95	2.2524	3.2502
26	0.0001	0.0008	0.0078	0.0726 15	0.4502 65	1.2832 95	2.2624	3.2602
27	0.0001	0.0008	0.0080	0.0742 16	0.4567 65	1.2927 95	2.2723	3.2702
28	0.0001	0.0008	0.0082	0.0757 16	0.4632 66	1.3022 95	2.2823	3.2802
29	0.0001	0.0008	0.0084	0.0774 16	0.4698 66	1.3117 95	2.2922	3.2902
30	0.0001	0.0009	0.0086	0.0790 17	0.4764 67	1.3212 95	2.3022	3.3002
31	0.0001	0.0009	0.0088	0.0807 17	0.4831 67	1.3308 95	2.3121	3.3102
32	0.0001	0.0009	0.0090	0.0824 17	0.4899 68	1.3403 95	2.3221	3.3202
33	0.0001	0.0009	0.0092	0.0841 18	0.4966 68	1.3499 96	2.3320	3.3302
34	0.0001	0.0009	0.0094	0.0859 18	0.5035 69	1.3594 96	2.3420	3.3402
35	0.0001	0.0010	0.0096	0.0877 18	0.5104 69	1.3690 96	2.3519	3.3502
36	0.0001	0.0010	0.0098	0.0896 19	0.5173 70	1.3786 96	2.3619	3.3602
37	0.0001	0.0010	0.0101	0.0915 19	0.5243 70	1.3881 96	2.3718	3.3702
38	0.0001	0.0010	0.0103	0.0934 19	0.5313 71	1.3977 96	2.3818	3.3802
39	0.0001	0.0011	0.0105	0.0953 20	0.5384 71	1.4073 96	2.3918	3.3902
40	0.0001	0.0011	0.0108	0.0973 20	0.5455 72	1.4170 96	2.4017	3.4002
41	0.0001	0.0011	0.0110	0.0993 20	0.5527 72	1.4266 96	2.4117	3.4102
42	0.0001	0.0011	0.0113	0.1014 21	0.5599 72	1.4362 96	2.4216	3.4202
43	0.0001	0.0012	0.0115	0.1035 21	0.5672 73	1.4458 96	2.4316	3.4302
44	0.0001	0.0012	0.0118	0.1057 22	0.5745 73	1.4555 96	2.4416	3.4402
45	0.0001	0.0012	0.0121	0.1078 22	0.5819 74	1.4651 97	2.4515	3.4502
46	0.0001	0.0013	0.0123	0.1101 22	0.5893 74	1.4748 97	2.4615	3.4602
47	0.0001	0.0013	0.0126	0.1123 23	0.5967 75	1.4845 97	2.4715	3.4701
48	0.0001	0.0013	0.0129	0.1146 23	0.6042 75	1.4941 97	2.4814	3.4801
49	0.0001	0.0013	0.0132	0.1169 24	0.6118 76	1.5038 97	2.4914	3.4901
50	0.0001	0.0014	0.0135	0.1193 24	0.6193 76	1.5135 97	2.5014	3.5001

Logarithms of Sums and Differences.

A	6.	7.	8.	9.	0.	1.	2.	3.
50	0.0001	0.0014	0.0135	0.1193 24	0.6193 76	1.5135 97	2.5014	3.5001
51	0.0001	0.0014	0.0138	0.1218 24	0.6269 76	1.5232 97	2.5113	3.5101
52	0.0001	0.0014	0.0141	0.1242 25	0.6346 77	1.5329 97	2.5213	3.5201
53	0.0001	0.0015	0.0145	0.1267 25	0.6423 77	1.5426 97	2.5313	3.5301
54	0.0002	0.0015	0.0148	0.1293 26	0.6501 78	1.5523 97	2.5413	3.5401
55	0.0002	0.0015	0.0151	0.1319 26	0.6578 78	1.5621 97	2.5512	3.5501
56	0.0002	0.0016	0.0155	0.1345 27	0.6657 78	1.5718 97	2.5612	3.5601
57	0.0002	0.0016	0.0158	0.1372 27	0.6735 79	1.5815 97	2.5712	3.5701
58	0.0002	0.0016	0.0162	0.1399 28	0.6814 79	1.5913 97	2.5811	3.5801
59	0.0002	0.0017	0.0166	0.1427 28	0.6893 80	1.6010 97	2.5911	3.5901
60	0.0002	0.0017	0.0170	0.1455 28	0.6973 80	1.6108 98	2.6011	3.6001
61	0.0002	0.0018	0.0173	0.1484 29	0.7053 80	1.6205 98	2.6111	3.6101
62	0.0002	0.0018	0.0177	0.1513 29	0.7134 81	1.6303 98	2.6210	3.6201
63	0.0002	0.0018	0.0181	0.1543 30	0.7215 81	1.6401 98	2.6310	3.6301
64	0.0002	0.0019	0.0186	0.1573 30	0.7296 81	1.6498 98	2.6410	3.6401
65	0.0002	0.0019	0.0190	0.1604 31	0.7377 82	1.6596 98	2.6510	3.6501
66	0.0002	0.0020	0.0194	0.1635 31	0.7459 82	1.6694 98	2.6609	3.6601
67	0.0002	0.0020	0.0199	0.1666 32	0.7541 82	1.6792 98	2.6709	3.6701
68	0.0002	0.0021	0.0203	0.1699 32	0.7624 83	1.6890 98	2.6809	3.6801
69	0.0002	0.0021	0.0208	0.1731 33	0.7707 83	1.6988 98	2.6909	3.6901
70	0.0002	0.0022	0.0212	0.1764 33	0.7790 83	1.7086 98	2.7009	3.7001
71	0.0002	0.0022	0.0217	0.1798 34	0.7874 84	1.7184 98	2.7108	3.7101
72	0.0002	0.0023	0.0222	0.1832 34	0.7957 84	1.7282 98	2.7208	3.7201
73	0.0002	0.0023	0.0227	0.1867 35	0.8042 84	1.7380 98	2.7308	3.7301
74	0.0002	0.0024	0.0232	0.1902 35	0.8126 85	1.7478 98	2.7408	3.7401
75	0.0002	0.0024	0.0238	0.1938 36	0.8211 85	1.7577 98	2.7508	3.7501
76	0.0002	0.0025	0.0243	0.1974 37	0.8296 85	1.7675 98	2.7608	3.7601
77	0.0003	0.0025	0.0248	0.2011 37	0.8381 85	1.7773 98	2.7707	3.7701
78	0.0003	0.0026	0.0254	0.2048 38	0.8467 86	1.7871 98	2.7807	3.7801
79	0.0003	0.0027	0.0260	0.2086 38	0.8553 86	1.7970 98	2.7907	3.7901
80	0.0003	0.0027	0.0266	0.2124 39	0.8639 86	1.8068 98	2.8007	3.8001
81	0.0003	0.0028	0.0272	0.2163 39	0.8725 87	1.8167 98	2.8107	3.8101
82	0.0003	0.0029	0.0278	0.2203 40	0.8812 87	1.8265 99	2.8207	3.8201
83	0.0003	0.0029	0.0284	0.2243 40	0.8899 87	1.8364 99	2.8306	3.8301
84	0.0003	0.0030	0.0291	0.2284 41	0.8986 87	1.8462 99	2.8406	3.8401
85	0.0003	0.0031	0.0297	0.2325 41	0.9074 88	1.8561 99	2.8506	3.8501
86	0.0003	0.0031	0.0304	0.2366 42	0.9162 88	1.8660 99	2.8606	3.8601
87	0.0003	0.0032	0.0311	0.2409 43	0.9250 88	1.8758 99	2.8706	3.8701
88	0.0003	0.0033	0.0318	0.2452 43	0.9338 88	1.8857 99	2.8806	3.8801
89	0.0003	0.0034	0.0325	0.2495 44	0.9426 89	1.8956 99	2.8906	3.8901
90	0.0003	0.0034	0.0332	0.2539 44	0.9515 89	1.9054 99	2.9005	3.9001
91	0.0004	0.0035	0.0339	0.2584 45	0.9604 89	1.9153 99	2.9105	3.9101
92	0.0004	0.0036	0.0347	0.2629 45	0.9693 89	1.9252 99	2.9205	3.9201
93	0.0004	0.0037	0.0355	0.2674 46	0.9782 89	1.9351 99	2.9305	3.9301
94	0.0004	0.0038	0.0363	0.2721 47	0.9872 90	1.9450 99	2.9405	3.9400
95	0.0004	0.0039	0.0371	0.2767 47	0.9962 90	1.9548 99	2.9505	3.9500
96	0.0004	0.0039	0.0379	0.2815 48	1.0052 90	1.9647 99	2.9605	3.9600
97	0.0004	0.0040	0.0387	0.2863 48	1.0142 90	1.9746 99	2.9705	3.9700
98	0.0004	0.0041	0.0396	0.2911 49	1.0232 91	1.9845 99	2.9805	3.9800
99	0.0004	0.0042	0.0405	0.2961 49	1.0323 91	1.9944 99	2.9904	3.9900
00	0.0004	0.0043	0.0414	0.3010 50	1.0414 91	2.0043 99	3.0004	4.0000

φ	0° lsin ltn	lsc	1° lsin ltn	lsc	2° lsin ltn	lsc	
00′	— ∞	00	8.2419 19 72	01	8.5428 31 36	03	60′
01′	6.4637 37	00	8.2490 91 71	01	8.5464 67 36	03	59′
02′	6.7648 48	00	8.2561 62 70	01	8.5500 03 36	03	58′
03′	6.9408 08	00	8.2630 31 69	01	8.5535 38 35	03	57′
04′	7.0658 58	00	8.2699*00 68	01	8.5571 73 35	03	56′
05′	7.1627 27	00	8.2766 67 67	01	8.5605 08 35	03	55′
06′	7.2419 19	00	8.2832 33 66	01	8.5640 43 34	03	54′
07′	7.3088 88	00	8.2898 99 65	01	8.5674 77 34	03	53′
08′	7.3668 68	00	8.2962 63 64	01	8.5708 11 34	03	52′
09′	7.4180 80	00	8.3025 26 63	01	8.5742 45 34	03	51′
10′	7.4637 37	00	8.3088 89 62	01	8.5776 79 33	03	50′
11′	7.5051 51	00	8.3150 50 61	01	8.5809 12 33	03	49′
12′	7.5429 29	00	8.3210 11 60	01	8.5842 45 33	03	48′
13′	7.5777 77	00	8.3270 71 59	01	8.5875 78 33	03	47′
14′	7.6099 99	00	8.3329 30 59	01	8.5907 11 32	03	46′
15′	7.6398 98	00	8.3388 89 58	01	8.5939 43 32	03	45′
16′	7.6678 78	00	8.3445 46 57	01	8.5972 75 32	03	44′
17′	7.6942 42	00	8.3502 03 56	01	8.6003 07 32	03	43′
18′	7.7190 90	00	8.3558 59 56	01	8.6035 38 31	04	42′
19′	7.7425 25	00	8.3613 14 55	01	8.6066 70 31	04	41′
20′	7.7648 48	00	8.3668 69 54	01	8.6097*01 31	04	40′
21′	7.7859 60	00	8.3722 23 54	01	8.6128 32 31	04	39′
22′	7.8061 62	00	8.3775 76 53	01	8.6159 63 31	04	38′
23′	7.8255 55	00	8.3828 29 52	01	8.6189 93 30	04	37′
24′	7.8439 39	00	8.3880 81 52	01	8.6220 23 30	04	36′
25′	7.8617 17	00	8.3931 32 51	01	8.6250 54 30	04	35′
26′	7.8787 87	00	8.3982 83 51	01	8.6279 83 30	04	34′
27′	7.8951 51	00	8.4032 33 50	01	8.6309 13 30	04	33′
28′	7.9109 09	00	8.4082 83 49	01	8.6339 43 29	04	32′
29′	7.9261 61	00	8.4131 32 49	01	8.6368 72 29	04	31′
30′	7.9408 09	00	8.4179 81 48	01	8.6397*01 29	04	30′
31′	7.9551 51	00	8.4227 29 48	02	8.6426 30 29	04	29′
32′	7.9689 89	00	8.4275 76 47	02	8.6454 59 29	04	28′
33′	7.9822 23 132	00	8.4322 23 47	02	8.6483 87 28	04	27′
34′	7.9952 52 128	00	8.4368 70 46	02	8.6511 15 28	04	26′
35′	8.0078 78 124	00	8.4414 16 46	02	8.6539 44 28	04	25′
36′	8.0200 00 121	00	8.4459 61 45	02	8.6567 71 28	04	24′
37′	8.0319 19 117	00	8.4504 06 45	02	8.6595 99 28	05	23′
38′	8.0435 35 114	00	8.4549 51 44	02	8.6622 27 27	05	22′
39′	8.0548 48 111	00	8.4593 95 44	02	8.6650 54 27	05	21′
40′	8.0658 58 109	00	8.4637 38 43	02	8.6677 82 27	05	20′
41′	8.0765 65 106	00	8.4680 82 43	02	8.6704 09 27	05	19′
42′	8.0870 70 103	00	8.4723 25 43	02	8.6731 36 27	05	18′
43′	8.0972 72 101	00	8.4765 67 42	02	8.6758 62 27	05	17′
44′	8.1072 72 99	00	8.4807 09 42	02	8.6784 89 26	05	16′
45′	8.1169 70 97	00	8.4848 51 41	02	8.6810 15 26	05	15′
46′	8.1265 65 94	00	8.4890 92 41	02	8.6837 42 26	05	14′
47′	8.1358 59 92	00	8.4930 33 41	02	8.6863 68 26	05	13′
48′	8.1450 50 90	00	8.4971 73 40	02	8.6889 94 26	05	12′
49′	8.1539 40 89	00	8.5011 13 40	02	8.6914 20 26	05	11′
50′	8.1627 27 87	00	8.5050 53 39	02	8.6940 45 26	05	10′
51′	8.1713 13 85	00	8.5090 92 39	02	8.6965 71 25	05	09′
52′	8.1797 98 84	00	8.5129 31 39	02	8.6991 96 25	05	08′
53′	8.1880 80 82	01	8.5167 70 38	02	8.7016 21 25	06	07′
54′	8.1961 62 80	01	8.5206 08 38	02	8.7041 46 25	06	06′
55′	8.2041 41 79	01	8.5243 46 38	02	8.7066 71 25	06	05′
56′	8.2119 20 78	01	8.5281 83 37	02	8.7090 96 25	06	04′
57′	8.2196 96 76	01	8.5318 21 37	03	8.7115 21 25	06	03′
58′	8.2271 72 75	01	8.5355 58 37	03	8.7140 45 24	06	02′
59′	8.2346 46 74	01	8.5392 94 37	03	8.7164 70 24	06	01′
60′	8.2419 19 72	01	8.5428 31 36	03	8.7188 94 24	06	00′
	89° lcos lctn	lcsc	88° lcos lctn	lcsc	87° lcos lctn	lcsc	θ

Logarithms of Circular Functions.

φ	3° l sin l tn		l sc	4° l sin l tn		l sc	5° l sin l tn		l sc	
00′	8.7188 94	24	06	8.8436 46	18	11	8.9403 20	14	17	**60′**
01′	8.7212 18	24	06	8.8454 65	18	11	8.9417 34	14	17	59′
02′	8.7236 42	24	06	8.8472 83	18	11	8.9432 49	14	17	58′
03′	8.7260 66	24	06	8.8490*01	18	11	8.9446 63	14	17	57′
04′	8.7283 90	24	06	8.8508 18	18	11	8.9460 77	14	17	56′
05′	8.7307 13	23	06	8.8525 36	18	11	8.9475 92	14	17	**55′**
06′	8.7330 37	23	06	8.8543 54	18	11	8.9489*06	14	17	54′
07′	8.7354 60	23	06	8.8560 72	18	11	8.9503 20	14	17	53′
08′	8.7377 83	23	06	8.8578 89	18	11	8.9517 34	14	17	52′
09′	8.7400 06	23	07	8.8595*07	17	11	8.9531 49	14	18	51′
10′	8.7423 29	23	07	8.8613 24	17	11	8.9545 63	14	18	**50′**
11′	8.7445 52	23	07	8.8630 42	17	12	8.9559 77	14	18	49′
12′	8.7468 75	23	07	8.8647 59	17	12	8.9573 91	14	18	48′
13′	8.7491 97	23	07	8.8665 76	17	12	8.9587*05	14	18	47′
14′	8.7513 20	22	07	8.8682 94	17	12	8.9601 19	14	18	46′
15′	8.7535 42	22	07	8.8699*11	17	12	8.9614 33	14	18	**45′**
16′	8.7557 65	22	07	8.8716 28	17	12	8.9628 46	14	18	44′
17′	8.7580 87	22	07	8.8733 45	17	12	8.9642 60	14	18	43′
18′	8.7602 09	22	07	8.8749 62	17	12	8.9655 74	14	19	42′
19′	8.7623 31	22	07	8.8766 78	17	12	8.9669 88	14	19	41′
20′	8.7645 52	22	07	8.8783 95	17	12	8.9682*01	14	19	**40′**
21′	8.7667 74	22	07	8.8799*12	17	13	8.9696*15	14	19	39′
22′	8.7688 96	22	08	8.8816 29	17	13	8.9709 29	14	19	38′
23′	8.7710 17	21	08	8.8833 45	17	13	8.9723 42	13	19	37′
24′	8.7731 39	21	08	8.8849 62	16	13	8.9736 56	13	19	36′
25′	8.7752 60	21	08	8.8865 78	16	13	8.9750 69	13	19	**35′**
26′	8.7773 81	21	08	8.8882 95	16	13	8.9763 82	13	20	34′
27′	8.7794*02	21	08	8.8898*11	16	13	8.9776 96	13	20	33′
28′	8.7815 23	21	08	8.8914 27	16	13	8.9789*09	13	20	32′
29′	8.7836 44	21	08	8.8930 44	16	13	8.9803 23	13	20	31′
30′	8.7857 65	21	08	8.8946 60	16	13	8.9816 36	13	20	**30′**
31′	8.7877 86	21	08	8.8962 76	16	14	8.9829 49	13	20	29′
32′	8.7898*06	20	08	8.8978 92	16	14	8.9842 62	13	20	28′
33′	8.7918 27	20	08	8.8994*08	16	14	8.9855 75	13	20	27′
34′	8.7939 47	20	08	8.9010 24	16	14	8.9868 88	13	21	26′
35′	8.7959 67	20	08	8.9026 40	16	14	8.9881*01	13	21	**25′**
36′	8.7979 88	20	09	8.9042 56	16	14	8.9894*15	13	21	24′
37′	8.7999*08	20	09	8.9057 71	16	14	8.9907 28	13	21	23′
38′	8.8019 28	20	09	8.9073 87	16	14	8.9919 40	13	21	22′
39′	8.8039 48	20	09	8.9089*03	16	14	8.9932 53	13	21	21′
40′	8.8059 67	20	09	8.9104 18	16	14	8.9945 66	13	21	**20′**
41′	8.8078 87	20	09	8.9119 34	15	15	8.9958 79	13	21	19′
42′	8.8098*07	20	09	8.9135 50	15	15	8.9970 92	13	22	18′
43′	8.8117 26	19	09	8.9150 65	15	15	8.9983*05	13	22	17′
44′	8.8137 46	19	09	8.9166 80	15	15	8.9996*17	13	22	16′
45′	8.8156 65	19	09	8.9181 96	15	15	9.0008 30	13	22	**15′**
46′	8.8175 85	19	09	8.9196*11	15	15	9.0021 43	13	22	14′
47′	8.8194*04	19	09	8.9211 26	15	15	9.0033 55	13	22	13′
48′	8.8213 23	19	10	8.9226 41	15	15	9.0046 68	12	22	12′
49′	8.8232 42	19	10	8.9241 56	15	15	9.0058 80	12	22	11′
50′	8.8251 61	19	10	8.9256 72	15	15	9.0070 93	12	23	**10′**
51′	8.8270 80	19	10	8.9271 87	15	16	9.0083*05	12	23	09′
52′	8.8289 99	19	10	8.9286*02	15	16	9.0095*18	12	23	08′
53′	8.8307 17	19	10	8.9301 16	15	16	9.0107 30	12	23	07′
54′	8.8326 36	19	10	8.9315 31	15	16	9.0120 43	12	23	06′
55′	8.8345 55	19	10	8.9330 46	15	16	9.0132 55	12	23	**05′**
56′	8.8363 73	18	10	8.9345 61	15	16	9.0144 67	12	23	04′
57′	8.8381 92	18	10	8.9359 76	15	16	9.0156 80	12	23	03′
58′	8.8400 10	18	10	8.9374 90	15	16	9.0168 92	12	24	02′
59′	8.8418 28	18	11	8.9388*05	15	16	9.0180*04	12	24	01′
60′	8.8436 46	18	11	8.9403 20	14	17	9.0192*16	12	24	**00′**
	86° l cos l ctn		l csc	85° l cos l ctn		l csc	84° l cos l ctn		l csc	θ

φ	S	l sin φ
	6.46	
0°00′.000		− ∞
	37	
1°51′.479		8.5108
	36	
2°49′.567		8.6929
	35	
3°32′.313		8.7904
	34	
4°07′.789		8.8574
	33	
4°38′.783		8.9085
	32	
5°06′.659		8.9498
	31	
5°32′.201		8.9845
	30	
5°55′.913		9.0143
	29	
6°18′.138		9.0405

φ	T	l tan φ
	6.46	
0°00′.000		− ∞
	37	
0°44′.155		8.1087
	38	
1°40′.555		8.4663
	39	
2°15′.168		8.5948
	40	
2°42′.563		8.6751
	41	
3°05′.959		8.7336
	42	
3°26′.717		8.7796
	43	
3°45′.567		8.8176
	44	
4°02′.954		8.8500
	45	
4°19′.171		8.8781
	46	
4°34′.427		8.9031
	47	
4°48′.875		8.9255
	48	
5°02′.628		8.9458
	49	
5°15′.780		8.9643
	50	
5°28′.401		8.9815
	51	
5°40′.550		8.9973
	52	
5°52′.278		9.0121
	53	
6°03′.623		9.0260

φ	l sec φ
	0.00
0°00′.000	
	00
0°52′.164	
	01
1°30′.348	
	02
1°56′.634	
	03
2°17′.998	
	04
2°36′.469	
	05
2°52′.976	
	06
3°08′.038	
	07
3°21′.977	
	08
3°35′.016	
	09
3°47′.300	
	10
3°58′.955	
	11
4°10′.064	
	12
4°20′.701	

φ	l sec φ
	0.00
4°20′.701	
	13
4°30′.918	
	14
4°40′.762	
	15
4°50′.271	
	16
4°59′.477	
	17
5°08′.407	
	18
5°17′.084	
	19
5°25′.528	
	20
5°33′.758	
	21
5°41′.789	
	22
5°49′.633	
	23
5°57′.305	
	24
6°04′.814	
	25
6°12′.170	

φ	l sin φ		l csc φ	l tan φ		l ctn φ	l sec φ		l cos φ	
6° 00′	9.0192	120	0.9808	9.0216	122	0.9784	0.0024	1	9.9976	**84° 00′**
6° 10′	9.0311	117	0.9689	9.0336	118	0.9664	0.0025	1	9.9975	83° 50′
6° 20′	9.0426	114	0.9574	9.0453	115	0.9547	0.0027	1	9.9973	83° 40′
6° 30′	9.0539	111	0.9461	9.0567	112	0.9433	0.0028	1	9.9972	83° 30′
6° 40′	9.0648	108	0.9352	9.0678	110	0.9322	0.0029	1	9.9971	83° 20′
6° 50′	9.0755	105	0.9245	9.0786	107	0.9214	0.0031	2	9.9969	83° 10′
7° 00′	9.0859	103	0.9141	9.0891	104	0.9109	0.0032	2	9.9968	**83° 00′**
7° 10′	9.0961	100	0.9039	9.0995	102	0.9005	0.0034	2	9.9966	82° 50′
7° 20′	9.1060	98	0.8940	9.1096	100	0.8904	0.0036	2	9.9964	82° 40′
7° 30′	9.1157	96	0.8843	9.1194	98	0.8806	0.0037	2	9.9963	82° 30′
7° 40′	9.1252	94	0.8748	9.1291	96	0.8709	0.0039	2	9.9961	82° 20′
7° 50′	9.1345	92	0.8655	9.1385	94	0.8615	0.0041	2	9.9959	82° 10′
8° 00′	9.1436	90	0.8564	9.1478	92	0.8522	0.0042	2	9.9958	**82° 00′**
8° 10′	9.1525	88	0.8475	9.1569	90	0.8431	0.0044	2	9.9956	81° 50′
8° 20′	9.1612	86	0.8388	9.1658	88	0.8342	0.0046	2	9.9954	81° 40′
8° 30′	9.1697	85	0.8303	9.1745	86	0.8255	0.0048	2	9.9952	81° 30′
8° 40′	9.1781	83	0.8219	9.1831	85	0.8169	0.0050	2	9.9950	81° 20′
8° 50′	9.1863	81	0.8137	9.1915	83	0.8085	0.0052	2	9.9948	81° 10′
9° 00′	9.1943	80	0.8057	9.1997	82	0.8003	0.0054	2	9.9946	**81° 00′**
9° 10′	9.2022	78	0.7978	9.2078	80	0.7922	0.0056	2	9.9944	80° 50′
9° 20′	9.2100	77	0.7900	9.2158	79	0.7842	0.0058	2	9.9942	80° 40′
9° 30′	9.2176	75	0.7824	9.2236	78	0.7764	0.0060	2	9.9940	80° 30′
9° 40′	9.2251	74	0.7749	9.2313	76	0.7687	0.0062	2	9.9938	80° 20′
9° 50′	9.2324	73	0.7676	9.2389	75	0.7611	0.0064	2	9.9936	80° 10′
10° 00′	9.2397	72	0.7603	9.2463	74	0.7537	0.0066	2	9.9934	**80° 00′**
	l cos θ		l sec θ	l ctn θ		l tan θ	l csc θ		l sin θ	θ

Logarithms of Circular Functions.

φ	l sin φ		l csc φ	l tan φ		l ctn φ	l sec φ		l cos φ	
10° 00′	9.2397	72	0.7603	9.2463	74	0.7537	0.0066	2	9.9934	**80° 00′**
10° 10′	9.2468	70	0.7532	9.2536	73	0.7464	0.0069	2	9.9931	79° 50′
10° 20′	9.2538	69	0.7462	9.2609	72	0.7391	0.0071	2	9.9929	79° 40′
10° 30′	9.2606	68	0.7394	9.2680	71	0.7320	0.0073	2	9.9927	79° 30′
10° 40′	9.2674	67	0.7326	9.2750	69	0.7250	0.0076	2	9.9924	79° 20′
10° 50′	9.2740	66	0.7260	9.2819	68	0.7181	0.0078	2	9.9922	79° 10′
11° 00′	9.2806	65	0.7194	9.2887	67	0.7113	0.0081	2	9.9919	**79° 00′**
11° 10′	9.2870	64	0.7130	9.2953	66	0.7047	0.0083	2	9.9917	78° 50′
11° 20′	9.2934	63	0.7066	9.3020	66	0.6980	0.0086	3	9.9914	78° 40′
11° 30′	9.2997	62	0.7003	9.3085	65	0.6915	0.0088	3	9.9912	78° 30′
11° 40′	9.3058	61	0.6942	9.3149	64	0.6851	0.0091	3	9.9909	78° 20′
11° 50′	9.3119	60	0.6881	9.3212	63	0.6788	0.0093	3	9.9907	78° 10′
12° 00′	9.3179	59	0.6821	9.3275	62	0.6725	0.0096	3	9.9904	**78° 00′**
12° 10′	9.3238	59	0.6762	9.3336	61	0.6664	0.0099	3	9.9901	77° 50′
12° 20′	9.3296	58	0.6704	9.3397	61	0.6603	0.0101	3	9.9899	77° 40′
12° 30′	9.3353	57	0.6647	9.3458	60	0.6542	0.0104	3	9.9896	77° 30′
12° 40′	9.3410	56	0.6590	9.3517	59	0.6483	0.0107	3	9.9893	77° 20′
12° 50′	9.3466	55	0.6534	9.3576	58	0.6424	0.0110	3	9.9890	77° 10′
13° 00′	9.3521	55	0.6479	9.3634	58	0.6366	0.0113	3	9.9887	**77° 00′**
13° 10′	9.3575	54	0.6425	9.3691	57	0.6309	0.0116	3	9.9884	76° 50′
13° 20′	9.3629	53	0.6371	9.3748	56	0.6252	0.0119	3	9.9881	76° 40′
13° 30′	9.3682	53	0.6318	9.3804	56	0.6196	0.0122	3	9.9878	76° 30′
13° 40′	9.3734	52	0.6266	9.3859	55	0.6141	0.0125	3	9.9875	76° 20′
13° 50′	9.3786	51	0.6214	9.3914	54	0.6086	0.0128	3	9.9872	76° 10′
14° 00′	9.3837	51	0.6163	9.3968	54	0.6032	0.0131	3	9.9869	**76° 00′**
14° 10′	9.3887	50	0.6113	9.4021	53	0.5979	0.0134	3	9.9866	75° 50′
14° 20′	9.3937	49	0.6063	9.4074	53	0.5926	0.0137	3	9.9863	75° 40′
14° 30′	9.3986	49	0.6014	9.4127	52	0.5873	0.0141	3	9.9859	75° 30′
14° 40′	9.4035	48	0.5965	9.4178	52	0.5822	0.0144	3	9.9856	75° 20′
14° 50′	9.4083	48	0.5917	9.4230	51	0.5770	0.0147	3	9.9853	75° 10′
15° 00′	9.4130	47	0.5870	9.4281	51	0.5719	0.0151	3	9.9849	**75° 00′**
15° 10′	9.4177	47	0.5823	9.4331	50	0.5669	0.0154	3	9.9846	74° 50′
15° 20′	9.4223	46	0.5777	9.4381	50	0.5619	0.0157	3	9.9843	74° 40′
15° 30′	9.4269	46	0.5731	9.4430	49	0.5570	0.0161	4	9.9839	74° 30′
15° 40′	9.4314	45	0.5686	9.4479	49	0.5521	0.0164	4	9.9836	74° 20′
15° 50′	9.4359	45	0.5641	9.4527	48	0.5473	0.0168	4	9.9832	74° 10′
16° 00′	9.4403	44	0.5597	9.4575	48	0.5425	0.0172	4	9.9828	**74° 00′**
16° 10′	9.4447	44	0.5553	9.4622	47	0.5378	0.0175	4	9.9825	73° 50′
16° 20′	9.4491	43	0.5509	9.4669	47	0.5331	0.0179	4	9.9821	73° 40′
16° 30′	9.4533	43	0.5467	9.4716	46	0.5284	0.0183	4	9.9817	73° 30′
16° 40′	9.4576	42	0.5424	9.4762	46	0.5238	0.0186	4	9.9814	73° 20′
16° 50′	9.4618	42	0.5382	9.4808	46	0.5192	0.0190	4	9.9810	73° 10′
17° 00′	9.4659	41	0.5341	9.4853	45	0.5147	0.0194	4	9.9806	**73° 00′**
	l cos θ		l sec θ	l ctn θ		l tan θ	l csc θ		l sin θ	θ

φ	l sin φ		l csc φ	l tan φ		l ctn φ	l sec φ		l cos φ	
17° 00′	9.4659	41	0.5341	9.4853	45	0.5147	0.0194	4	9.9806	**73° 00′**
17° 10′	9.4700	41	0.5300	9.4898	45	0.5102	0.0198	4	9.9802	72° 50′
17° 20′	9.4741	40	0.5259	9.4943	44	0.5057	0.0202	4	9.9798	72° 40′
17° 30′	9.4781	40	0.5219	9.4987	44	0.5013	0.0206	4	9.9794	72° 30′
17° 40′	9.4821	40	0.5179	9.5031	44	0.4969	0.0210	4	9.9790	72° 20′
17° 50′	9.4861	39	0.5139	9.5075	43	0.4925	0.0214	4	9.9786	72° 10′
18° 00′	9.4900	39	0.5100	9.5118	43	0.4882	0.0218	4	9.9782	**72° 00′**
18° 10′	9.4939	38	0.5061	9.5161	43	0.4839	0.0222	4	9.9778	71° 50′
18° 20′	9.4977	38	0.5023	9.5203	42	0.4797	0.0226	4	9.9774	71° 40′
18° 30′	9.5015	38	0.4985	9.5245	42	0.4755	0.0230	4	9.9770	71° 30′
18° 40′	9.5052	37	0.4948	9.5287	42	0.4713	0.0235	4	9.9765	71° 20′
18° 50′	9.5090	37	0.4910	9.5329	41	0.4671	0.0239	4	9.9761	71° 10′
19° 00′	9.5126	37	0.4874	9.5370	41	0.4630	0.0243	4	9.9757	**71° 00′**
19° 10′	9.5163	36	0.4837	9.5411	41	0.4589	0.0248	4	9.9752	70° 50′
19° 20′	9.5199	36	0.4801	9.5451	40	0.4549	0.0252	4	9.9748	70° 40′
19° 30′	9.5235	36	0.4765	9.5491	40	0.4509	0.0257	4	9.9743	70° 30′
19° 40′	9.5270	35	0.4730	9.5531	40	0.4469	0.0261	5	9.9739	70° 20′
19° 50′	9.5306	35	0.4694	9.5571	40	0.4429	0.0266	5	9.9734	70° 10′
20° 00′	9.5341	35	0.4659	9.5611	39	0.4389	0.0270	5	9.9730	**70° 00′**
20° 10′	9.5375	34	0.4625	9.5650	39	0.4350	0.0275	5	9.9725	69° 50′
20° 20′	9.5409	34	0.4591	9.5689	39	0.4311	0.0279	5	9.9721	69° 40′
20° 30′	9.5443	34	0.4557	9.5727	39	0.4273	0.0284	5	9.9716	69° 30′
20° 40′	9.5477	33	0.4523	9.5766	38	0.4234	0.0289	5	9.9711	69° 20′
20° 50′	9.5510	33	0.4490	9.5804	38	0.4196	0.0294	5	9.9706	69° 10′
21° 00′	9.5543	33	0.4457	9.5842	38	0.4158	0.0298	5	9.9702	**69° 00′**
21° 10′	9.5576	33	0.4424	9.5879	38	0.4121	0.0303	5	9.9697	68° 50′
21° 20′	9.5609	32	0.4391	9.5917	37	0.4083	0.0308	5	9.9692	68° 40′
21° 30′	9.5641	32	0.4359	9.5954	37	0.4046	0.0313	5	9.9687	68° 30′
21° 40′	9.5673	32	0.4327	9.5991	37	0.4009	0.0318	5	9.9682	68° 20′
21° 50′	9.5704	32	0.4296	9.6028	37	0.3972	0.0323	5	9.9677	68° 10′
22° 00′	9.5736	31	0.4264	9.6064	36	0.3936	0.0328	5	9.9672	**68° 00′**
22° 10′	9.5767	31	0.4233	9.6100	36	0.3900	0.0333	5	9.9667	67° 50′
22° 20′	9.5798	31	0.4202	9.6136	36	0.3864	0.0339	5	9.9661	67° 40′
22° 30′	9.5828	30	0.4172	9.6172	36	0.3828	0.0344	5	9.9656	67 30′
22° 40′	9.5859	30	0.4141	9.6208	36	0.3792	0.0349	5	9.9651	67° 20′
22° 50′	9.5889	30	0.4111	9.6243	35	0.3757	0.0354	5	9.9646	67° 10′
23° 00′	9.5919	30	0.4081	9.6279	35	0.3721	0.0360	5	9.9640	**67° 00′**
23° 10′	9.5948	30	0.4052	9.6314	35	0.3686	0.0365	5	9.9635	66° 50′
23° 20′	9.5978	29	0.4022	9.6348	35	0.3652	0.0371	5	9.9629	66° 40′
23° 30′	9.6007	29	0.3993	9.6383	35	0.3617	0.0376	5	9.9624	66° 30′
23° 40′	9.6036	29	0.3964	9.6417	34	0.3583	0.0382	6	9.9618	66° 20′
23° 50′	9.6065	29	0.3935	9.6452	34	0.3548	0.0387	6	9.9613	66° 10′
24° 00′	9.6093	28	0.3907	9.6486	34	0.3514	0.0393	6	9.9607	**66° 00′**
	l cos θ		l sec θ	l ctn θ		l tan θ	l csc θ		l sin θ	θ

Logarithms of Circular Functions.

φ	l sin φ		l csc φ	l tan φ		l ctn φ	l sec φ		l cos φ	
24° 00′	9.6093	28	0.3907	9.6486	34	0.3514	0.0393	6	9.9607	**66° 00′**
24° 10′	9.6121	28	0.3879	9.6520	34	0.3480	0.0398	6	9.9602	65° 50′
24° 20′	9.6149	28	0.3851	9.6553	34	0.3447	0.0404	6	9.9596	65° 40′
24° 30′	9.6177	28	0.3823	9.6587	33	0.3413	0.0410	6	9.9590	65° 30′
24° 40′	9.6205	28	0.3795	9.6620	33	0.3380	0.0416	6	9.9584	65° 20′
24° 50′	9.6232	27	0.3768	9.6654	33	0.3346	0.0421	6	9.9579	65° 10′
25° 00′	9.6259	27	0.3741	9.6687	33	0.3313	0.0427	6	9.9573	**65° 00′**
25° 10′	9.6286	27	0.3714	9.6720	33	0.3280	0.0433	6	9.9567	64° 50′
25° 20′	9.6313	27	0.3687	9.6752	33	0.3248	0.0439	6	9.9561	64° 40′
25° 30′	9.6340	26	0.3660	9.6785	33	0.3215	0.0445	6	9.9555	64° 30′
25° 40′	9.6366	26	0.3634	9.6817	32	0.3183	0.0451	6	9.9549	64° 20′
25° 50′	9.6392	26	0.3608	9.6850	32	0.3150	0.0457	6	9.9543	64° 10′
26° 00′	9.6418	26	0.3582	9.6882	32	0.3118	0.0463	6	9.9537	**64° 00′**
26° 10′	9.6444	26	0.3556	9.6914	32	0.3086	0.0470	6	9.9530	63° 50′
26° 20′	9.6470	26	0.3530	9.6946	32	0.3054	0.0476	6	9.9524	63° 40′
26° 30′	9.6495	25	0.3505	9.6977	32	0.3023	0.0482	6	9.9518	63° 30′
26° 40′	9.6521	25	0.3479	9.7009	31	0.2991	0.0488	6	9.9512	63° 20′
26° 50′	9.6546	25	0.3454	9.7040	31	0.2960	0.0495	6	9.9505	63° 10′
27° 00′	9.6570	25	0.3430	9.7072	31	0.2928	0.0501	6	9.9499	**63° 00′**
27° 10′	9.6595	25	0.3405	9.7103	31	0.2897	0.0508	6	9.9492	62° 50′
27° 20′	9.6620	24	0.3380	9.7134	31	0.2866	0.0514	7	9.9486	62° 40′
27° 30′	9.6644	24	0.3356	9.7165	31	0.2835	0.0521	7	9.9479	62° 30′
27° 40′	9.6668	24	0.3332	9.7196	31	0.2804	0.0527	7	9.9473	62° 20′
27° 50′	9.6692	24	0.3308	9.7226	31	0.2774	0.0534	7	9.9466	62° 10′
28° 00′	9.6716	24	0.3284	9.7257	30	0.2743	0.0541	7	9.9459	**62° 00′**
28° 10′	9.6740	24	0.3260	9.7287	30	0.2713	0.0547	7	9.9453	61° 50′
28° 20′	9.6763	23	0.3237	9.7317	30	0.2683	0.0554	7	9.9446	61° 40′
28° 30′	9.6787	23	0.3213	9.7348	30	0.2652	0.0561	7	9.9439	61° 30′
28° 40′	9.6810	23	0.3190	9.7378	30	0.2622	0.0568	7	9.9432	61° 20′
28° 50′	9.6833	23	0.3167	9.7408	30	0.2592	0.0575	7	9.9425	61° 10′
29° 00′	9.6856	23	0.3144	9.7438	30	0.2562	0.0582	7	9.9418	**61° 00′**
29° 10′	9.6878	23	0.3122	9.7467	30	0.2533	0.0589	7	9.9411	60° 50′
29° 20′	9.6901	22	0.3099	9.7497	30	0.2503	0.0596	7	9.9404	60° 40′
29° 30′	9.6923	22	0.3077	9.7526	29	0.2474	0.0603	7	9.9397	60° 30′
29° 40′	9.6946	22	0.3054	9.7556	29	0.2444	0.0610	7	9.9390	60° 20′
29° 50′	9.6968	22	0.3032	9.7585	29	0.2415	0.0617	7	9.9383	60° 10′
30° 00′	9.6990	22	0.3010	9.7614	29	0.2386	0.0625	7	9.9375	**60° 00′**
30° 10′	9.7012	22	0.2988	9.7644	29	0.2356	0.0632	7	9.9368	59° 50′
30° 20′	9.7033	22	0.2967	9.7673	29	0.2327	0.0639	7	9.9361	59° 40′
30° 30′	9.7055	21	0.2945	9.7701	29	0.2299	0.0647	7	9.9353	59° 30′
30° 40′	9.7076	21	0.2924	9.7730	29	0.2270	0.0654	7	9.9346	59° 20′
30° 50′	9.7097	21	0.2903	9.7759	29	0.2241	0.0662	8	9.9338	59° 10′
31° 00′	9.7118	21	0.2882	9.7788	29	0.2212	0.0669	8	9.9331	**59° 00′**
	l cos θ		l sec θ	l ctn θ		l tan θ	l csc θ		l sin θ	θ

Logarithms of Circular Functions.

φ	l sin φ		l csc φ	l tan φ		l ctn φ	l sec φ		l cos φ	
31° 00′	9.7118	21	0.2882	9.7788	29	0.2212	0.0669	8	9.9331	**59° 00′**
31° 10′	9.7139	21	0.2861	9.7816	29	0.2184	0.0677	8	9.9323	58° 50′
31° 20′	9.7160	21	0.2840	9.7845	28	0.2155	0.0685	8	9.9315	58° 40′
31° 30′	9.7181	21	0.2819	9.7873	28	0.2127	0.0692	8	9.9308	58° 30′
31° 40′	9.7201	20	0.2799	9.7902	28	0.2098	0.0700	8	9.9300	58° 20′
31° 50′	9.7222	20	0.2778	9.7930	28	0.2070	0.0708	8	9.9292	58° 10′
32° 00′	9.7242	20	0.2758	9.7958	28	0.2042	0.0716	8	9.9284	**58° 00′**
32° 10′	9.7262	20	0.2738	9.7986	28	0.2014	0.0724	8	9.9276	57° 50′
32° 20′	9.7282	20	0.2718	9.8014	28	0.1986	0.0732	8	9.9268	57° 40′
32° 30′	9.7302	20	0.2698	9.8042	28	0.1958	0.0740	8	9.9260	57° 30′
32° 40′	9.7322	20	0.2678	9.8070	28	0.1930	0.0748	8	9.9252	57° 20′
32° 50′	9.7342	20	0.2658	9.8097	28	0.1903	0.0756	8	9.9244	57° 10′
33° 00′	9.7361	19	0.2639	9.8125	28	0.1875	0.0764	8	9.9236	**57° 00′**
33° 10′	9.7380	19	0.2620	9.8153	28	0.1847	0.0772	8	9.9228	56° 50′
33° 20′	9.7400	19	0.2600	9.8180	28	0.1820	0.0781	8	9.9219	56° 40′
33° 30′	9.7419	19	0.2581	9.8208	27	0.1792	0.0789	8	9.9211	56° 30′
33° 40′	9.7438	19	0.2562	9.8235	27	0.1765	0.0797	8	9.9203	56° 20′
33° 50′	9.7457	19	0.2543	9.8263	27	0.1737	0.0806	8	9.9194	56° 10′
34° 00′	9.7476	19	0.2524	9.8290	27	0.1710	0.0814	9	9.9186	**56° 00′**
34° 10′	9.7494	19	0.2506	9.8317	27	0.1683	0.0823	9	9.9177	55° 50′
34° 20′	9.7513	18	0.2487	9.8344	27	0.1656	0.0831	9	9.9169	55° 40′
34° 30′	9.7531	18	0.2469	9.8371	27	0.1629	0.0840	9	9.9160	55° 30′
34° 40′	9.7550	18	0.2450	9.8398	27	0.1602	0.0849	9	9.9151	55° 20′
34° 50′	9.7568	18	0.2432	9.8425	27	0.1575	0.0858	9	9.9142	55° 10′
35° 00′	9.7586	18	0.2414	9.8452	27	0.1548	0.0866	9	9.9134	**55° 00′**
35° 10′	9.7604	18	0.2396	9.8479	27	0.1521	0.0875	9	9.9125	54° 50′
35° 20′	9.7622	18	0.2378	9.8506	27	0.1494	0.0884	9	9.9116	54° 40′
35° 30′	9.7640	18	0.2360	9.8533	27	0.1467	0.0893	9	9.9107	54° 30′
35° 40′	9.7657	18	0.2343	9.8559	27	0.1441	0.0902	9	9.9098	54° 20′
35° 50′	9.7675	17	0.2325	9.8586	27	0.1414	0.0911	9	9.9089	54° 10′
36° 00′	9.7692	17	0.2308	9.8613	27	0.1387	0.0920	9	9.9080	**54° 00′**
36° 10′	9.7710	17	0.2290	9.8639	27	0.1361	0.0930	9	9.9070	53° 50′
36° 20′	9.7727	17	0.2273	9.8666	26	0.1334	0.0939	9	9.9061	53° 40′
36° 30′	9.7744	17	0.2256	9.8692	26	0.1308	0.0948	9	9.9052	53° 30′
36° 40′	9.7761	17	0.2239	9.8718	26	0.1282	0.0958	9	9.9042	53° 20′
36° 50′	9.7778	17	0.2222	9.8745	26	0.1255	0.0967	9	9.9033	53° 10′
37° 00′	9.7795	17	0.2205	9.8771	26	0.1229	0.0977	10	9.9023	**53° 00′**
37° 10′	9.7811	17	0.2189	9.8797	26	0.1203	0.0986	10	9.9014	52° 50′
37° 20′	9.7828	17	0.2172	9.8824	26	0.1176	0.0996	10	9.9004	52° 40′
37° 30′	9.7844	16	0.2156	9.8850	26	0.1150	0.1005	10	9.8995	52° 30′
37° 40′	9.7861	16	0.2139	9.8876	26	0.1124	0.1015	10	9.8985	52° 20′
37° 50′	9.7877	16	0.2123	9.8902	26	0.1098	0.1025	10	9.8975	52° 10′
38° 00′	9.7893	16	0.2107	9.8928	26	0.1072	0.1035	10	9.8965	**52° 00′**
	l cos θ		l sec θ	l ctn θ		l tan θ	l csc θ		l sin θ	θ

φ	l sin φ		l csc φ	l tan φ		l ctn φ	l sec φ		l cos φ	
38° 00′	9.7893	16	0.2107	9.8928	26	0.1072	0.1035	10	9.8965	**52° 00′**
38° 10′	9.7910	16	0.2090	9.8954	26	0.1046	0.1045	10	9.8955	51° 50′
38° 20′	9.7926	16	0.2074	9.8980	26	0.1020	0.1055	10	9.8945	51° 40′
38° 30′	9.7941	16	0.2059	9.9006	26	0.0994	0.1065	10	9.8935	51° 30′
38° 40′	9.7957	16	0.2043	9.9032	26	0.0968	0.1075	10	9.8925	51° 20′
38° 50′	9.7973	16	0.2027	9.9058	26	0.0942	0.1085	10	9.8915	51° 10′
39° 00′	9.7989	16	0.2011	9.9084	26	0.0916	0.1095	10	9.8905	**51° 00′**
39° 10′	9.8004	16	0.1996	9.9110	26	0.0890	0.1105	10	9.8895	50° 50′
39° 20′	9.8020	15	0.1980	9.9135	26	0.0865	0.1116	10	9.8884	50° 40′
39° 30′	9.8035	15	0.1965	9.9161	26	0.0839	0.1126	10	9.8874	50° 30′
39° 40′	9.8050	15	0.1950	9.9187	26	0.0813	0.1136	10	9.8864	50° 20′
39° 50′	9.8066	15	0.1934	9.9212	26	0.0788	0.1147	11	9.8853	50° 10′
40° 00′	9.8081	15	0.1919	9.9238	26	0.0762	0.1157	11	9.8843	**50° 00′**
40° 10′	9.8096	15	0.1904	9.9264	26	0.0736	0.1168	11	9.8832	49° 50′
40° 20′	9.8111	15	0.1889	9.9289	26	0.0711	0.1179	11	9.8821	49° 40′
40° 30′	9.8125	15	0.1875	9.9315	26	0.0685	0.1190	11	9.8810	49° 30′
40° 40′	9.8140	15	0.1860	9.9341	26	0.0659	0.1200	11	9.8800	49° 20′
40° 50′	9.8155	15	0.1845	9.9366	26	0.0634	0.1211	11	9.8789	49° 10′
41° 00′	9.8169	15	0.1831	9.9392	26	0.0608	0.1222	11	9.8778	**49° 00′**
41° 10′	9.8184	14	0.1816	9.9417	25	0.0583	0.1233	11	9.8767	48° 50′
41° 20′	9.8198	14	0.1802	9.9443	25	0.0557	0.1244	11	9.8756	48° 40′
41° 30′	9.8213	14	0.1787	9.9468	25	0.0532	0.1255	11	9.8745	48° 30′
41° 40′	9.8227	14	0.1773	9.9494	25	0.0506	0.1267	11	9.8733	48° 20′
41° 50′	9.8241	14	0.1759	9.9519	25	0.0481	0.1278	11	9.8722	48° 10′
42° 00′	9.8255	14	0.1745	9.9544	25	0.0456	0.1289	11	9.8711	**48° 00′**
42° 10′	9.8269	14	0.1731	9.9570	25	0.0430	0.1301	11	9.8699	47° 50′
42° 20′	9.8283	14	0.1717	9.9595	25	0.0405	0.1312	12	9.8688	47° 40′
42° 30′	9.8297	14	0.1703	9.9621	25	0.0379	0.1324	12	9.8676	47° 30′
42° 40′	9.8311	14	0.1689	9.9646	25	0.0354	0.1335	12	9.8665	47° 20′
42° 50′	9.8324	14	0.1676	9.9671	25	0.0329	0.1347	12	9.8653	47° 10′
43° 00′	9.8338	14	0.1662	9.9697	25	0.0303	0.1359	12	9.8641	**47° 00′**
43° 10′	9.8351	13	0.1649	9.9722	25	0.0278	0.1371	12	9.8629	46° 50′
43° 20′	9.8365	13	0.1635	9.9747	25	0.0253	0.1382	12	9.8618	46° 40′
43° 30′	9.8378	13	0.1622	9.9772	25	0.0228	0.1394	12	9.8606	46° 30′
43° 40′	9.8391	13	0.1609	9.9798	25	0.0202	0.1406	12	9.8594	46° 20′
43° 50′	9.8405	13	0.1595	9.9823	25	0.0177	0.1418	12	9.8582	46° 10′
44° 00′	9.8418	13	0.1582	9.9848	25	0.0152	0.1431	12	9.8569	**46° 00′**
44° 10′	9.8431	13	0.1569	9.9874	25	0.0126	0.1443	12	9.8557	45° 50′
44° 20′	9.8444	13	0.1556	9.9899	25	0.0101	0.1455	12	9.8545	45° 40′
44° 30′	9.8457	13	0.1543	9.9924	25	0.0076	0.1468	12	9.8532	45° 30′
44° 40′	9.8469	13	0.1531	9.9949	25	0.0051	0.1480	12	9.8520	45° 20′
44° 50′	9.8482	13	0.1518	9.9975	25	0.0025	0.1493	13	9.8507	45° 10′
45° 00′	9.8495	13	0.1505	0.0000	25	0.0000	0.1505	13	9.8495	**45° 00′**
	l cos θ		l sec θ	l ctn θ		l tan θ	l csc θ		l sin θ	θ

Inverse Circular Functions.

log u	$\sin^{-1}u$		$\cos^{-1}u$	$\tan^{-1}u$		$\text{ctn}^{-1}u$	log u	$\sin^{-1}u$		$\cos^{-1}u$	$\tan^{-1}u$		$\text{ctn}^{-1}u$
9.	°		°	°		°	9.	°		°	°		°
00	5.74	13	84.26	5.71	13	84.29	**50**	18.43	44	71.57	17.55	38	72.45
01	5.87	14	84.13	5.84	13	84.16	51	18.88	45	71.12	17.93	39	72.07
02	6.01	14	83.99	5.98	14	84.02	52	19.34	46	70.66	18.32	39	71.68
03	6.15	14	83.85	6.12	14	83.88	53	19.81	48	70.19	18.72	40	71.28
04	6.30	15	83.70	6.26	14	83.74	54	20.29	49	69.71	19.12	41	70.88
05	6.44	15	83.56	6.40	15	83.60	**55**	20.78	50	69.22	19.54	42	70.46
06	6.59	15	83.41	6.55	15	83.45	56	21.29	51	68.71	19.95	42	70.05
07	6.75	16	83.25	6.70	15	83.30	57	21.81	53	68.19	20.38	43	69.62
08	6.91	16	83.09	6.86	16	83.14	58	22.35	54	67.65	20.82	44	69.18
09	7.07	16	82.93	7.01	16	82.99	59	22.90	56	67.10	21.26	45	68.74
10	7.23	17	82.77	7.18	16	82.82	**60**	23.46	57	66.54	21.71	45	68.29
11	7.40	17	82.60	7.34	17	82.66	61	24.04	59	65.96	22.17	46	67.83
12	7.58	18	82.42	7.51	17	82.49	62	24.64	61	65.36	22.63	47	67.37
13	7.75	18	82.25	7.68	17	82.32	63	25.25	62	64.75	23.10	48	69.90
14	7.93	18	82.07	7.86	18	82.14	64	25.88	64	64.12	23.58	48	66.42
15	8.12	19	81.88	8.04	18	81.96	**65**	26.53	66	63.47	24.07	49	65.93
16	8.31	19	81.69	8.22	19	81.78	66	27.20	68	62.80	24.56	50	65.44
17	8.51	20	81.49	8.41	19	81.59	67	27.89	70	62.11	25.07	51	64.93
18	8.71	20	81.29	8.61	20	81.39	68	28.60	72	61.40	25.58	51	64.42
19	8.91	21	81.09	8.80	20	81.20	69	29.33	74	60.67	26.09	52	63.91
20	9.12	21	80.88	9.01	20	80.99	**70**	30.08	76	59.92	26.62	53	63.38
21	9.33	22	80.67	9.21	21	80.79	71	30.85	79	59.15	27.15	54	62.85
22	9.55	22	80.45	9.42	21	80.58	72	31.66	81	58.34	27.69	54	62.31
23	9.78	23	80.22	9.64	22	80.36	73	32.48	84	57.52	28.24	55	61.76
24	10.01	23	79.99	9.86	22	80.14	74	33.34	87	56.66	28.79	56	61.21
25	10.24	24	79.76	10.08	23	79.92	**75**	34.22	90	55.78	29.35	56	60.65
26	10.48	24	79.52	10.31	23	79.69	76	35.13		54.87	29.92	57	60.08
27	10.73	25	79.27	10.55	24	79.45	77	36.07		53.93	30.49	58	59.51
28	10.98	26	79.02	10.79	24	79.21	78	37.05		52.95	31.07	58	58.93
29	11.24	26	78.76	11.03	25	78.97	79	38.07		51.93	31.66	59	58.34
30	11.51	27	78.49	11.28	25	78.72	**80**	39.12		50.88	32.25	60	57.75
31	11.78	28	78.22	11.54	26	78.46	81	40.21		49.79	32.85	60	57.15
32	12.06	28	77.94	11.80	26	78.20	82	41.35		48.65	33.45	61	56.55
33	12.34	29	77.66	12.07	27	77.93	83	42.54		47.46	34.06	61	55.94
34	12.64	30	77.36	12.34	28	77.66	84	43.78		46.22	34.68	62	55.32
35	12.94	30	77.06	12.62	28	77.38	**85**	45.07		44.93	35.30	62	54.70
36	13.24	31	76.76	12.90	29	77.10	86	46.42		43.58	35.92	63	54.08
37	13.56	32	76.44	13.19	29	76.81	87	47.84		42.16	36.55	63	53.45
38	13.88	33	76.12	13.49	30	76.51	88	49.34		40.66	37.18	64	52.82
39	14.21	33	75.79	13.79	31	76.21	89	50.92		39.08	37.82	64	52.18
40	14.55	34	75.45	14.10	31	75.90	**90**	52.59		37.41	38.46	64	51.54
41	14.89	35	75.11	14.42	32	75.58	91	54.37		35.63	39.11	65	50.89
42	15.25	36	74.75	14.74	32	75.26	92	56.28		33.72	39.75	65	50.25
44	15.61	37	74.39	15.06	33	74.94	93	58.34		31.66	40.40	65	49.60
45	15.99	38	74.01	15.40	34	74.60	94	60.57		29.43	41.05	65	48.95
46	16.37	39	73.63	15.74	34	74.26	**95**	63.03		26.97	41.71	66	48.29
47	16.76	40	73.24	16.09	35	73.91	96	65.78		24.22	42.37	66	47.63
48	17.16	41	72.84	16.44	36	73.56	97	68.95		21.05	43.02	66	46.98
49	7.58	42	72.42	16.80	37	73.20	98	72.74		17.26	43.68	66	46.32
50	18.00	43	72.00	17.17	37	72.83	99	77.75		12.25	44.34	66	45.66
			71.57	17.55	38	72.45	**00**	90.00		00.00	45.00	66	45.00

Inverse Circular Functions.

$\log u$	$\sin^{-1} u$		$\cos^{-1} u$	$\log u$	$\sin^{-1} u$		$\cos^{-1} u$	$\log u$	$\sin^{-1} u$		$\cos^{-1} u$
9.	°		°	**9.**	°		°	**9.**	°		°
750	34.22	9	55.78	**800**	39.12	11	50.88	**850**	45.07	13	44.93
751	34.31	9	55.69	801	39.23	11	50.77	851	45.20	13	44.80
752	34.40	9	55.60	802	39.34	11	50.66	852	45.33	13	44.67
753	34.49	9	55.51	803	39.44	11	50.56	853	45.47	13	44.53
754	34.58	9	55.42	804	39.55	11	50.45	854	45.60	13	44.40
755	34.67	9	55.33	**805**	39.66	11	50.34	**855**	45.74	14	44.26
756	34.76	9	55.24	806	39.77	11	50.23	856	45.87	14	44.13
757	34.85	9	55.15	807	39.88	11	50.12	857	46.01	14	43.99
758	34.95	9	55.05	808	39.99	11	50.01	858	46.15	14	43.85
759	35.04	9	54.96	809	40.10	11	49.90	859	46.28	14	43.72
760	35.13	9	54.87	**810**	40.21	11	49.79	**860**	46.42	14	43.58
761	35.22	9	54.78	811	40.33	11	49.67	861	46.56	14	43.44
762	35.32	9	54.68	812	40.44	11	49.56	862	46.70	14	43.30
763	35.41	9	54.59	813	40.55	11	49.45	863	46.84	14	43.16
764	35.50	9	54.50	814	40.66	11	49.34	864	46.98	14	43.02
765	35.60	9	54.40	**815**	40.78	11	49.22	**865**	47.12	14	42.88
766	35.69	9	54.31	816	40.89	11	49.11	866	47.27	14	42.73
767	35.79	10	54.21	817	41.01	11	48.99	867	47.41	14	42.59
768	35.88	10	54.12	818	41.12	12	48.88	868	47.55	14	42.45
769	35.98	10	54.02	819	41.24	12	48.76	869	47.70	14	42.30
770	36.07	10	53.93	**820**	41.35	12	48.65	**870**	47.84	15	42.16
771	36.17	10	53.83	821	41.47	12	48.53	871	47.99	15	42.01
772	36.27	10	53.73	822	41.59	12	48.41	872	48.14	15	41.86
773	36.36	10	53.64	823	41.70	12	48.30	873	48.28	15	41.72
774	36.46	10	53.54	824	41.82	12	48.18	874	48.43	15	41.57
775	36.56	10	53.44	**825**	41.94	12	48.06	**875**	48.58	15	41.42
776	36.66	10	53.34	826	42.06	12	47.94	876	48.73	15	41.27
777	36.76	10	53.24	827	42.18	12	47.82	877	48.88	15	41.12
778	36.85	10	53.15	828	42.30	12	47.70	878	49.03	15	40.97
779	36.95	10	53.05	829	42.42	12	47.58	879	49.19	15	40.81
780	37.05	10	52.95	**830**	42.54	12	47.46	**880**	49.34	15	40.66
781	37.15	10	52.85	831	42.66	12	47.34	881	49.49	15	40.51
782	37.25	10	52.75	832	42.78	12	47.22	882	49.65	16	40.35
783	37.35	10	52.65	833	42.90	12	47.10	883	49.80	16	40.20
784	37.45	10	52.55	834	43.03	12	46.97	884	49.96	16	40.04
785	37.56	10	52.44	**835**	43.15	12	46.85	**885**	50.12	16	39.88
786	37.66	10	52.34	836	43.27	12	46.73	886	50.28	16	39.72
787	37.76	10	52.24	837	43.40	12	46.60	887	50.44	16	39.56
788	37.86	10	52.14	838	43.52	13	46.48	888	50.60	16	39.40
789	37.96	10	52.04	839	43.65	13	46.35	889	50.76	16	39.24
790	38.07	10	51.93	**840**	43.78	13	46.22	**890**	50.92	16	39.08
791	38.17	10	51.83	841	43.90	13	46.10	891	51.08	16	38.92
792	38.28	10	51.72	842	44.03	13	45.97	892	51.25	16	38.75
793	38.38	10	51.62	843	44.16	13	45.84	893	51.41	17	38.59
794	38.48	10	51.52	844	44.29	13	45.71	894	51.58	17	38.42
795	38.59	11	51.41	**845**	44.41	13	45.59	**895**	51.74	17	38.26
796	38.69	11	51.31	846	44.54	13	45.46	896	51.91	17	38.09
797	38.80	11	51.20	847	44.67	13	45.33	897	52.08	17	37.92
798	38.91	11	51.09	848	44.80	13	45.20	898	52.25	17	37.75
799	39.01	11	50.99	849	44.94	13	45.06	899	52.42	17	37.58
800	39.12	11	50.88	**850**	45.07	13	44.93	**900**	52.59	17	37.41

log u	$\sin^{-1}u$		$\cos^{-1}u$	log u	$\sin^{-1}u$		$\cos^{-1}u$	log u	$\sin^{-1}u$	log u	$\sin^{-1}u$
9.	°		°	**9.**	°		°	**9.**	°	**9.**	°
900	52.59	17	37.41	**950**	63.03	26	26.97	**9900**	77.75	**9950**	81.32
901	52.76	17	37.24	951	63.29	26	26.71	9901	77.81	9951	81.41
902	52.94	17	37.06	952	63.56	27	26.44	9902	77.87	9952	81.50
903	53.11	18	36.89	953	63.82	27	26.18	9903	77.94	9953	81.59
904	53.29	18	36.71	954	64.09	27	25.91	9904	78.00	9954	81.68
905	53.47	18	36.53	**955**	64.37	27	25.63	**9905**	78.06	**9955**	81.77
906	53.65	18	36.35	956	64.64	28	25.36	9906	78.12	9956	81.86
907	53.83	18	36.17	957	64.92	28	25.08	9907	78.18	9957	81.95
908	54.01	18	35.99	958	65.21	29	24.79	9908	78.25	9958	82.04
909	54.19	18	35.81	959	65.49	29	24.51	9909	78.31	9959	82.14
910	54.37	18	35.63	**960**	65.78	29	24.22	**9910**	78.38	**9960**	82.24
911	54.56	19	35.44	961	66.08	30	23.92	9911	78.44	9961	82.33
912	54.74	19	35.26	962	66.38	30	23.62	9912	78.50	9962	82.43
913	54.93	19	35.07	963	66.68	31	23.32	9913	78.57	9963	82.5
914	55.12	19	34.88	964	66.99	31	23.01	9914	78.64	9964	82.6
915	55.31	19	34.69	**965**	67.30	32	22.70	**9915**	78.70	**9965**	82.7
916	55.50	19	34.50	966	67.62	32	22.38	9916	78.77	9966	82.8
917	55.69	19	34.31	967	67.95	33	22.05	9917	78.83	9967	82.9
918	55.89	19	34.11	968	68.27	33	21.73	9918	78.90	9968	83.1
919	56.08	20	33.92	969	68.61	34	21.39	9919	78.97	9969	83.2
920	56.28	20	33.72	**970**	68.95	34	21.05	**9920**	79.04	**9970**	83.3
921	56.48	20	33.52	971	69.29	35	20.71	9921	79.10	9971	83.4
922	56.68	20	33.32	972	69.65	36	20.35	9922	79.17	9972	83.5
923	56.88	20	33.12	973	70.01	36	19.99	9923	79.24	9973	83.6
924	57.08	20	32.92	974	70.37	37	19.63	9924	79.31	9974	83.7
925	57.29	21	32.71	**975**	70.75	38	19.25	**9925**	79.38	**9975**	83.9
926	57.49	21	32.51	976	71.13	39	18.87	9926	79.45	9976	84.0
927	57.70	21	32.30	977	71.52	39	18.48	9927	79.52	9977	84.1
928	57.91	21	32.09	978	71.92	40	18.08	9928	79.60	9978	84.2
929	58.12	21	31.88	979	72.33	41	17.67	9929	79.67	9979	84.4
930	58.34	21	31.66	**980**	72.74	42	17.26	**9930**	79.74	**9980**	84.5
931	58.55	22	31.45	981	73.18	44	16.82	9931	79.81	9981	84.6
932	58.77	22	31.23	982	73.62	45	16.38	9932	79.89	9982	84.8
933	58.99	22	31.01	983	74.07	46	15.93	9933	79.96	9983	84.9
934	59.21	22	30.79	984	74.54	48	15.46	9934	80.04	9984	85.1
935	59.43	22	30.57	**985**	75.03	49	14.97	**9935**	80.11	**9985**	85.2
936	59.65	23	30.35	986	75.53	51	14.47	9936	80.19	9986	85.4
937	59.88	23	30.12	987	76.05	53	13.95	9937	80.26	9987	85.6
938	60.11	23	29.89	988	76.59	55	13.41	9938	80.34	9988	85.7
939	60.34	23	29.66	989	77.16	58	12.84	9939	80.42	9989	85.9
940	60.57	23	29.43	**990**	77.75	61	12.25	**9940**	80.50	**9990**	86.1
941	60.81	24	29.19	991	78.38		11.62	9941	80.58	9991	86.3
942	61.04	24	28.96	992	79.04		10.96	9942	80.66	9992	86.5
943	61.28	24	28.72	993	79.74		10.26	9943	80.74	9993	86.7
944	61.52	24	28.48	994	80.50		9.50	9944	80.82	9994	87.0
945	61.77	25	28.23	**995**	81.32		8.68	**9945**	80.90	**9995**	87.3
946	62.02	25	27.98	996	82.24		7.76	9946	80.98	9996	87.5
947	62.27	25	27.73	997	83.3		6.7	9947	81.07	9997	87.9
948	62.52	25	27.48	998	84.5		5.5	9948	81.15	9998	88.3
949	62.77	26	27.23	999	86.1		3.9	9949	81.24	9999	88.8
950	63.03	26	26.97	**000**	90.		0.	**9950**	81.32	**0000**	90.

Logarithms of Hyperbolic Functions.

x	gd *x*		*x*	gd *x*		*x*	l Sh *x*		l Ch *x*		l Th *x*	
	°			°			0.		0.		9.	
0.00	0.0000	5730	**0.50**	27.524	508	**1.00**	0701	57	1884	33	8817	24
0.01	0.5729	5729	0.51	28.031	506	1.01	0758	57	1917	33	8840	23
0.02	1.1458	5728	0.52	28.535	503	1.02	0815	56	1950	33	8864	23
0.03	1.7186	5727	0.53	29.037	501	1.03	0871	56	1984	34	8887	23
0.04	2.2912	5725	0.54	29.537	498	1.04	0927	56	2018	34	8909	22
0.05	2.8636	5722	**0.55**	30.034	496	**1.05**	0982	56	2051	34	8931	22
0.06	3.4357	5719	0.56	30.529	494	1.06	1038	55	2086	34	8952	21
0.07	4.0074	5716	0.57	31.021	491	1.07	1093	55	2120	34	8973	21
0.08	4.5788	5711	0.58	31.511	488	1.08	1148	55	2154	34	8994	20
0.09	5.1497	5706	0.50	31.998	486	1.09	1203	54	2189	35	9014	20
0.10	5.720	570	**0.60**	32.483	483	**1.10**	1257	54	2223	35	9034	19
0.11	6.290	570	0.61	32.965	481	1.11	1311	54	2258	35	9053	19
0.12	6.859	569	0.62	33.444	478	1.12	1365	54	2293	35	9072	19
0.13	7.428	568	0.63	33.921	475	1.13	1419	54	2328	35	9090	18
0.14	7.995	567	0.64	34.395	473	1.14	1472	53	2364	35	9108	18
0.15	8.562	567	**0.65**	34.867	470	**1.15**	1525	53	2399	36	9126	18
0.16	9.128	566	0.66	35.336	467	1.16	1578	53	2435	36	9144	17
0.17	9.694	565	0.67	35.802	465	1.17	1631	53	2470	36	9161	17
0.18	10.258	564	0.68	36.265	462	1.18	1684	52	2506	36	9177	17
0.19	10.821	563	0.69	36.726	459	1.19	1736	52	2542	36	9194	16
0.20	11.384	562	**0.70**	37.183	456	**1.20**	1788	52	2578	36	9210	16
0.21	11.945	561	0.71	37.638	454	1.21	1840	52	2615	36	9226	16
0.22	12.505	559	0.72	38.091	451	1.22	1892	52	2651	36	9241	15
0.23	13.063	558	0.73	38.540	448	1.23	1944	52	2688	37	9256	15
0.24	13.621	557	0.74	38.987	445	1.24	1995	51	2724	37	9271	15
0.25	14.177	556	**0.75**	39.431	443	**1.25**	2046	51	2761	37	9285	14
0.26	14.732	554	0.76	39.872	440	1.26	2098	51	2798	37	9300	14
0.27	15.285	553	0.77	40.310	437	1.27	2148	51	2835	37	9314	14
0.28	15.837	551	0.78	40.746	434	1.28	2199	51	2872	37	9327	14
0.29	16.388	550	0.79	41.179	431	1.29	2250	51	2909	37	9341	13
0.30	16.937	548	**0.80**	41.608	428	**1.30**	2300	50	2947	37	9354	13
0.31	17.484	546	0.81	42.035	426	1.31	2351	50	2984	38	9367	13
0.32	18.030	545	0.82	42.460	423	1.32	2401	50	3022	38	9379	12
0.33	18.573	543	0.83	42.881	420	1.33	2451	50	3059	38	9391	12
0.34	19.116	541	0.84	43.299	417	1.34	2501	50	3097	38	9404	12
0.35	19.656	540	**0.85**	43.715	414	**1.35**	2551	50	3135	38	9415	12
0.36	20.195	538	0.86	44.128	411	1.36	2600	50	3173	38	9427	11
0.37	20.732	536	0.87	44.537	408	1.37	2650	49	3211	38	9438	11
0.38	21.267	534	0.88	44.944	406	1.38	2699	49	3249	38	9450	11
0.39	21.800	532	0.89	45.348	403	1.39	2748	49	3288	38	9460	11
0.40	22.331	530	**0.90**	45.750	400	**1.40**	2797	49	3326	38	9471	11
0.41	22.859	528	0.91	46.148	397	1.41	2846	49	3365	39	9482	10
0.42	23.386	526	0.92	46.544	394	1.42	2895	49	3403	39	9492	10
0.43	23.911	524	0.93	46.936	391	1.43	2944	49	3442	39	9502	10
0.44	24.434	522	0.94	47.326	388	1.44	2993	49	3481	39	9512	10
0.45	24.955	519	**0.95**	47.713	386	**1.45**	3041	48	3520	39	9522	10
0.46	25.473	517	0.96	48.097	383	1.46	3090	48	3559	39	9531	9
0.47	25.989	515	0.97	48.478	380	1.47	3138	48	3598	39	9540	9
0.48	26.503	513	0.98	48.857	377	1.48	3186	48	3637	39	9549	9
0.49	27.015	510	0.99	49.232	374	1.49	3234	48	3676	39	9558	9
0.50	27.524	508	**1.00**	49.605	371	**1.50**	3282	49	3715	39	9567	9

x	l Sh x	l Ch x	l Th x	x	l Sh x	l Ch x	l Th x
	0.	0.	9.		0.	0.	9.
1.50	3282 48	3716 39	9567 9	**2.00**	5595 45	5754 42	9841 3
1.51	3330 48	3754 39	9576 8	2.01	5640 45	5796 42	9844 3
1.52	3378 48	3794 39	9584 8	2.02	5685 45	5838 42	9847 3
1.53	3426 48	3833 40	9592 8	2.03	5730 45	5880 42	9850 3
1.54	3474 48	3873 40	9601 8	2.04	5775 45	5922 42	9853 3
1.55	3521 48	3913 40	9608 8	**2.05**	5820 45	5964 42	9856 3
1.56	3569 47	3952 40	9616 8	2.06	5865 45	6006 42	9859 3
1.57	3616 47	3992 40	9624 8	2.07	5910 45	6048 42	9862 3
1.58	3663 47	4032 40	9631 7	2.08	5955 45	6090 42	9864 3
1.59	3711 47	4072 40	9639 7	2.09	6000 45	6132 42	9867 3
1.60	3758 47	4112 40	9646 7	**2.10**	6044 45	6175 42	9870 3
1.61	3805 47	4152 40	9653 7	2.11	6089 45	6217 42	9872 3
1.62	3852 47	4192 40	9660 7	2.12	6134 45	6259 42	9875 3
1.63	3899 47	4232 40	9666 7	2.13	6178 45	6301 42	9877 2
1.64	3946 47	4273 40	9673 7	2.14	6223 45	6343 42	9880 2
1.65	3992 47	4313 40	9679 6	**2.15**	6268 45	6386 42	9882 2
1.66	4039 47	4353 40	9686 6	2.16	6312 45	6428 42	9884 2
1.67	4086 47	4394 40	9692 6	2.17	6357 45	6470 42	9887 2
1.68	4132 47	4434 41	9698 6	2.18	6401 45	6512 42	9889 2
1.69	4179 46	4475 41	9704 6	2.19	6446 45	6555 42	9891 2
1.70	4225 46	4515 41	9710 6	**2.20**	6491 45	6597 42	9893 2
1.71	4272 46	4556 41	9716 6	2.21	6535 44	6640 42	9895 2
1.72	4318 46	4597 41	9721 6	2.22	6580 44	6682 42	9898 2
1.73	4364 46	4637 41	9727 5	2.23	6624 44	6724 42	9900 2
1.74	4411 46	4678 41	9732 5	2.24	6668 44	6767 42	9902 2
1.75	4457 46	4719 41	9738 5	**2.25**	6713 44	6809 42	9904 2
1.76	4503 46	4760 41	9743 5	2.26	6757 44	6852 42	9905 2
1.77	4549 46	4801 41	9748 5	2.27	6802 44	6894 43	9907 2
1.78	4595 46	4842 41	9753 5	2.28	6846 44	6937 43	9909 2
1.79	4641 46	4883 41	9758 5	2.29	6890 44	6979 43	9911 2
1.80	4687 46	4924 41	9763 5	**2.30**	6935 44	7022 43	9913 2
1.81	4733 46	4965 41	9767 5	2.31	6979 44	7064 43	9914 2
1.82	4778 46	5006 41	9772 5	2.32	7023 44	7107 43	9916 2
1.83	4824 46	5048 41	9776 4	2.33	7067 44	7150 43	9918 2
1.84	4870 46	5089 41	9781 4	2.34	7112 44	7192 43	9919 2
1.85	4915 46	5130 41	9785 4	**2.35**	7156 44	7235 43	9921 2
1.86	4961 46	5172 41	9789 4	2.36	7200 44	7278 43	9923 2
1.87	5007 46	5213 41	9794 4	2.37	7244 44	7320 43	9924 2
1.88	5052 46	5254 41	9798 4	2.38	7289 44	7363 43	9926 1
1.89	5098 45	5296 41	9802 4	2.39	7333 44	7406 43	9927 1
1.90	5143 45	5337 42	9806 4	**2.40**	7377 44	7448 43	9929 1
1.91	5188 45	5379 42	9810 4	2.41	7421 44	7491 43	9930 1
1.92	5234 45	5421 42	9813 4	2.42	7465 44	7534 43	9931 1
1.93	5279 45	5462 42	9817 4	2,43	7509 44	7577 43	9933 1
1.94	5324 45	5504 42	9821 4	2.44	7553 44	7619 43	9934 1
1.95	5370 45	5545 42	9824 4	**2.45**	7597 44	7662 43	9935 1
1.96	5415 45	5587 42	9828 3	2.46	7642 44	7705 43	9937 1
1.97	5460 45	5629 42	9831 3	2.47	7686 44	7748 43	9938 1
1.98	5505 45	5671 42	9834 3	2.48	7730 44	7791 43	9939 1
1.99	5550 45	5713 42	9838 3	2.49	7774 44	7833 43	9940 1
2.00	5595 45	5754 42	9841 3	**2.50**	7818 44	7876 43	9941 1

Logarithms of Hyperbolic Functions.

x	l Sh x	l Ch x	l Th x
	0.	**0.**	**9.**
2.50	7818 44	7876 43	9941
2.51	7862 44	7919 43	9943
2.52	7906 44	7962 43	9944
2.53	7950 44	8005 43	9945
2.54	7994 44	8048 43	9946
2.55	8038 44	8091 43	9947
2.56	8082 44	8134 43	9948
2.57	8126 44	8176 43	9949
2.58	8169 44	8219 43	9950
2.59	8213 44	8262 43	9951
2.60	8257 44	8305 43	9952
2.61	8301 44	8348 43	9953
2.62	8345 44	8391 43	9954
2.63	8389 44	8434 43	9955
2.64	8433 44	8477 43	9956
2.65	8477 44	8520 43	9957
2.66	8521 44	8563 43	9958
2.67	8564 44	8606 43	9958
2.68	8608 44	8649 43	9959
2.69	8652 44	8692 43	9960
2.70	8696 44	8735 43	9961
2.71	8740 44	8778 43	9962
2.72	8784 44	8821 43	9962
2.73	8827 44	8864 43	9963
2.74	8871 44	8907 43	9964
2.75	8915 44	8951 43	9965
2.76	8959 44	8994 43	9965
2.77	9003 44	9037 43	9966
2.78	9046 44	9080 43	9967
2.79	9090 44	9123 43	9967
2.80	9134 44	9166 43	9968
2.81	9178 44	9209 43	9969
2.82	9221 44	9252 43	9969
2.83	9265 44	9295 43	9970
2.84	9309 44	9338 43	9970
2.85	9353 44	9382 43	9971
2.86	9396 44	9425 43	9972
2.87	9440 44	9468 43	9972
2.88	9484 44	9511 43	9973
2.89	9527 44	9554 43	9973
2.90	9571 44	9597 43	9974
2.91	9615 44	9641 43	9974
2.92	9658 44	9684 43	9975
2.93	9702 44	9727 43	9975
2.94	9746 44	9770 43	9976
2.95	9789 44	9813 43	9976
2.96	9833 44	9856 43	9977
2.97	9877 44	9900 43	9977
2.98	9920 44	9943 43	9978
2.99	9964 44	9986 43	9978
3.00	1.0008 44	1.0029 43	9978

x	l Sh x	l Ch x	l Th x
3.0	1.0008 436	1.0029 432	9.9978
3.1	1.0444 436	1.0462 433	9.9982
3.2	1.0880 436	1.0894 433	9.9986
3.3	1.1316 435	1.1327 433	9.9988
3.4	1.1751 435	1.1761 433	9.9990
3.5	1.2186 435	1.2194 434	9.9992
3.6	1.2621 435	1.2628 434	9.9994
3.7	1.3056 435	1.3061 434	9.9995
3.8	1.3491 435	1.3495 434	9.9996
3.9	1.3925 435	1.3929 434	9.9996
4.0	1.4360 435	1.4363 434	9.9997
4.1	1.4795 435	1.4797 434	9.9998
4.2	1.5229 434	1.5231 434	9.9998
4.3	1.5664 434	1.5665 434	9.9998
4.4	1.6098 434	1.6099 434	9.9999
4.5	1.6532 434	1.6533 434	9.9999
4.6	1.6967 434	1.6968 434	9.9999
4.7	1.7401 434	1.7402 434	9.9999
4.8	1.7836 434	1.7836 434	9.9999
4.9	1.8270 434	1.8270 434	0.0000
5.0	1.8704 434	1.8705 434	0.0000
5.1	1.9139 434	1.9139 434	0.0000
5.2	1.9573 434	1.9573 434	0.0000
5.3	2.0007 434	2.0007 434	0.0000
5.4	2.0442 434	2.0442 434	0.0000
5.5	2.0876 434	2.0876 434	0.0000
5.6	2.1310 434	2.1310 434	0.0000
5.7	2.1744 434	2.1745 434	0.0000
5.8	2.2179 434	2.2179 434	0.0000
5.9	2.2613 434	2.2613 434	0.0000
6.0	2.3047 4343	2.3047 4343	0.0000
7.0	2.7390 4343	2.7390 4343	0.0000
8.0	3.1733 4343	3.1733 4343	0.0000
9.0	3.6076 4343	3.6076 4343	0.0000
10.0	4.0419 4343	4.0419 4343	0.0000

For higher values:

$\log \mathrm{Sh}\, x = \log \mathrm{Ch}\, x = x\mu - 0.301030;$

$\mathrm{Sh}^{-1} u = \mathrm{Ch}^{-1} u = (\log u + 0.3010)\, \mu^{-1}.$

n	$n\mu$	$n\mu^{-1}$	n
1	0.434294	2.302585	1
2	0.868589	4.605170	2
3	1.302883	6.907755	3
4	1.737178	9.210340	4
5	2.171472	11.512925	5
6	2.605767	13.815511	6
7	3.040061	16.118096	7
8	3.474356	18.420681	8
9	3.908650	20.723266	9
10	4.342945	23.025851	10

Natural Sines and Cosines.

ϕ	$\sin\phi$	$\cos\phi$	
0°	.000000	1.0000	**90°**
10′	.002909	1.0000	50′
20′	.005818	1.0000	40′
30′	.008727	1.0000	30′
40′	.011635	0.9999	20′
50′	.014544	0.9999	10′
1°	.017452	0.9998	**89°**
10′	.02036	0.9998	50′
20′	.02327	0.9997	40′
30′	.02618	0.9997	30′
40′	.02908	0.9996	20′
50′	.03199	0.9995	10′
2°	.03490	0.9994	**88°**
10′	.03781	0.9993	50′
20′	.04071	0.9992	40′
30′	.04362	0.9990	30′
40′	.04653	0.9989	20′
50′	.04943	0.9988	10′
3°	.05234	0.9986	**87°**
10′	.05524	0.9985	50′
20′	.05814	0.9983	40′
30′	.06105	0.9981	30′
40′	.06395	0.9980	20′
50′	.06685	0.9978	10′
4°	.06976	0.9976	**86°**
10′	.07266	0.9974	50′
20′	.07556	0.9971	40′
30′	.07846	0.9969	30′
40′	.08136	0.9967	20′
50′	.08426	0.9964	10′
5°	.08716	0.9962	**85°**
10′	.09005	0.9959	50′
20′	.09295	0.9957	40′
30′	.09585	0.9954	30′
40′	.09874	0.9951	20′
50′	.10164	0.9948	10′
6°	.10453	0.9945	**84°**
10′	.1074	0.9942	50′
20′	.1103	0.9939	40′
30′	.1132	0.9936	30′
40′	.1161	0.9932	20′
50′	.1190	0.9929	10′
7°	.1219	0.9925	**83°**
10′	.1248	0.9922	50′
20′	.1276	0.9918	40′
30′	.1305	0.9914	30′
			82°
	$\cos\theta$	$\sin\theta$	θ

ϕ	$\sin\phi$	$\cos\phi$	
7°			
30′	.1305	.9914	30′
40′	.1334	.9911	20′
50′	.1363	.9907	10′
8°	.1392	.9903	**82°**
10′	.1421	.9899	50′
20′	.1449	.9894	40′
30′	.1478	.9890	30′
40′	.1507	.9886	20′
50′	.1536	.9881	10′
9°	.1564	.9877	**81°**
10′	.1593	.9872	50′
20′	.1622	.9868	40′
30′	.1650	.9863	30′
40′	.1679	.9858	20′
50′	.1708	.9853	10′
10°	.1736	.9848	**80°**
10′	.1765	.9843	50′
20′	.1794	.9838	40′
30′	.1822	.9833	30′
40′	.1851	.9827	20′
50′	.1880	.9822	10′
11°	.1908	.9816	**79°**
10′	.1937	.9811	50′
20′	.1065	.9805	40′
30′	.1994	.9799	30′
40′	.2022	.9793	20′
50′	.2051	.9787	10′
12°	.2079	.9781	**78°**
10′	.2108	.9775	50′
20′	.2136	.9769	40′
30′	.2164	.9763	30′
40′	.2193	.9757	20′
50′	.2221	.9750	10′
13°	.2250	.9744	**77°**
10′	.2278	.9737	50′
20′	.2306	.9730	40′
30′	.2334	.9724	30′
40′	.2363	.9717	20′
50′	.2391	.9710	10′
14°	.2419	.9703	**76°**
10′	.2447	.9696	50′
20′	.2476	.9689	40′
30′	.2504	.9681	30′
40′	.2532	.9674	20′
50′	.2560	.9667	10′
15°	.2588	.9659	**75°**
	$\cos\theta$	$\sin\theta$	θ

ϕ	$\sin\phi$	$\cos\phi$	
15°	.2588	.9659	**75°**
10′	.2616	.9652	50′
20′	.2644	.9644	40′
30′	.2672	.9636	30′
40′	.2700	.9628	20′
50′	.2728	.9621	10′
16°	.2756	.9613	**74°**
10′	.2784	.9605	50′
20′	.2812	.9596	40′
30′	.2840	.9588	30′
40′	.2868	.9580	20′
50′	.2896	.9572	10′
17°	.2924	.9563	**73°**
10′	.2952	.9555	50′
20′	.2979	.9546	40′
30′	.3007	.9537	30′
40′	.3035	.9528	20′
50′	.3062	.9520	10′
18°	.3090	.9511	**72°**
10′	.3118	.9502	50′
20′	.3145	.9492	40′
30′	.3173	.9483	30′
40′	.3201	.9474	20′
50′	.3228	.9465	10′
19°	.3256	.9455	**71°**
10′	.3283	.9446	50′
20′	.3311	.9436	40′
30′	.3338	.9426	30′
40′	.3365	.9417	20′
50′	.3393	.9407	10′
20°	.3420	.9397	**70°**
10′	.3448	.9387	50′
20′	.3475	.9377	40′
30′	.3502	.9367	30′
40′	.3529	.9356	20′
50′	.3557	.9346	10′
21°	.3584	.9336	**69°**
10′	.3611	.9325	50′
20′	.3638	.9315	40′
30′	.3665	.9304	30′
40′	.3692	.9293	20′
50′	.3719	.9283	10′
22°	.3746	.9272	**68°**
10′	.3773	.9261	50′
20′	.3800	.9250	40′
30′	.3827	.9239	30′
			67°
	$\cos\theta$	$\sin\theta$	θ

Natural Sines and Cosines.

φ	sin φ	cos φ	
22°			
30′	.3827	.9239	30′
40′	.3854	.9228	20′
50′	.3881	.9216	10′
23°	.3907	.9205	**67°**
10′	.3934	.9194	50′
20′	.3961	.9182	40′
30′	.3987	.9171	30′
40′	.4014	.9159	20′
50′	.4041	.9147	10′
24°	.4067	.9135	**66°**
10′	.4094	.9124	50′
20′	.4120	.9112	40′
30′	.4147	.9100	30′
40′	.4173	.9088	20′
50′	.4200	.9075	10′
25°	.4226	.9063	**65°**
10′	.4253	.9051	50′
20′	.4279	.9038	40′
30′	.4305	.9026	30′
40′	.4331	.9013	20′
50′	.4358	.9001	10′
26°	.4384	.8988	**64°**
10′	.4410	.8975	50′
20′	.4436	.8962	40′
30′	.4462	.8949	30′
40′	.4488	.8936	20′
50′	.4514	.8923	10′
27°	.4540	.8910	**63°**
10′	.4566	.8897	50′
20′	.4592	.8884	40′
30′	.4617	.8870	30′
40′	.4643	.8857	20′
50′	.4669	.8843	10′
28°	.4695	.8829	**62°**
10′	.4720	.8816	50′
20′	.4746	.8802	40′
30′	.4772	.8788	30′
40′	.4797	.8774	20′
50′	.4823	.8760	10′
29°	.4848	.8746	**61°**
10′	.4874	.8732	50′
20′	.4899	.8718	40′
30′	.4924	.8704	30′
40′	.4950	.8689	20′
50′	.4975	.8675	10′
30°	.5000	.8660	**60°**
	cos θ	sin θ	θ

φ	sin φ	cos φ	
30°	.5000	.8660	**60°**
10′	.5025	.8646	50′
20′	.5050	.8631	40′
30′	.5075	.8616	30′
40′	.5100	.8601	20′
50′	.5125	.8587	10′
31°	.5150	.8572	**59°**
10′	.5175	.8557	50′
20′	.5200	.8542	40′
30′	.5225	.8526	30′
40′	.5250	.8511	20′
50′	.5275	.8496	10′
32°	.5299	.8480	**58°**
10′	.5324	.8465	50′
20′	.5348	.8450	40′
30′	.5373	.8434	30′
40′	.5398	.8418	20′
50′	.5422	.8403	10′
33°	.5446	.8387	**57°**
10′	.5471	.8371	50′
20′	.5495	.8355	40′
30′	.5519	.8339	30′
40′	.5544	.8323	20′
50′	.5568	.8307	10′
34°	.5592	.8290	**56°**
10′	.5616	.8274	50′
20′	.5640	.8258	40′
30′	.5664	.8241	30′
40′	.5688	.8225	20′
50′	.5712	.8208	10′
35°	.5736	.8192	**55°**
10′	.5760	.8175	50′
20′	.5783	.8158	40′
30′	.5807	.8141	30′
40′	.5831	.8124	20′
50′	.5854	.8107	10′
36°	.5878	.8090	**54°**
10′	.5901	.8073	50′
20′	.5925	.8056	40′
30′	.5948	.8039	30′
40′	.5972	.8021	20′
50′	.5995	.8004	10′
37°	.6018	.7986	**53°**
10′	.6041	.7969	50′
20′	.6065	.7951	40′
30′	.6088	.7934	30′
			52°
	cos θ	sin θ	θ

φ	sin φ	cos φ	
37°			
30′	.6088	.7934	30′
40′	.6111	.7916	20′
50′	.6134	.7898	10′
38°	.6157	.7880	**52°**
10′	.6180	.7862	50′
20′	.6202	.7844	40′
30′	.6225	.7826	30′
40′	.6248	.7808	20′
50′	.6271	.7790	10′
39°	.6293	.7771	**51°**
10′	.6316	.7753	50′
20′	.6338	.7735	40′
30′	.6361	.7716	30′
40′	.6383	.7698	20′
50′	.6406	.7679	10′
40°	.6428	.7660	**50°**
10′	.6450	.7642	50′
20′	.6472	.7623	40′
30′	.6494	.7604	30′
40′	.6517	.7585	20′
50′	.6539	.7566	10′
41°	.6561	.7547	**49°**
10′	.6583	.7528	50′
20′	.6604	.7509	40′
30′	.6626	.7490	30′
40′	.6648	.7470	20′
50′	.6670	.7451	10′
42°	.6691	.7431	**48°**
10′	.6713	.7412	50′
20′	.6734	.7392	40′
30′	.6756	.7373	30′
40′	.6777	.7353	20′
50′	.6799	.7333	10′
43°	.6820	.7314	**47°**
10′	.6841	.7294	50′
20′	.6862	.7274	40′
30′	.6884	.7254	30′
40′	.6905	.7234	20′
50′	.6926	.7214	10′
44°	.6947	.7193	**46°**
10′	.6967	.7173	50′
20′	.6988	.7153	40′
30′	.7009	.7133	30′
40′	.7030	.7112	20′
50′	.7050	.7092	10′
45°	.7071	.7071	**45°**
	cos θ	sin θ	θ

Natural Tangents and Cotangents.

φ	tan φ	ctn φ	
0°	.000000		**90°**
10′	.002909		50′
20′	.005818		40′
30′	.008727		30′
40′	.011636		20′
50′	.014545		10′
1°	.017455	57.	**89°**
10′	.02036	49.	50′
20′	.02328	43.	40′
30′	.02619	38.	30′
40′	.02910	34.	20′
50′	.03201	31.	10′
2°	.03492	28.6	**88°**
10′	.03783	26.4	50′
20′	.04075	24.5	40′
30′	.04366	22.9	30′
40′	.04658	21.5	20′
50′	.04949	20.2	10′
3°	.05241	19.1	**87°**
10′	.05533	18.1	50′
20′	.05824	17.2	40′
30′	.06116	16.3	30′
40′	.06408	15.6	20′
50′	.06700	14.9	10′
4°	.06993	14.3	**86°**
10′	.07285	13.73	50′
20′	.07578	13.20	40′
30′	.07870	12.71	30′
40′	.08163	12.25	20′
50′	.08456	11.83	10′
5°	.08749	11.43	**85°**
10′	.09042	11.06	50′
20′	.09335	10.71	40′
30′	.09629	10.39	30′
40′	.09923	10.08	20′
50′	.10216	9.79	10′
6°	.10510	9.51	**84°**
10′	.1080	9.26	50′
20′	.1110	9.01	40′
30′	.1139	8.78	30′
40′	.1169	8.56	20′
50′	.1198	8.34	10′
7°	.1228	8.14	**83°**
10′	.1257	7.95	50′
20′	.1287	7.77	40′
30′	.1317	7.60	30′
			82°
	ctn θ	tan θ	θ

φ	tan φ	ctn φ	
7°			
30′	.1317	7.60	30′
40′	.1346	7.43	20′
50′	.1376	7.27	10′
8°	.1405	7.12	**82°**
10′	.1435	6.97	50′
20′	.1465	6.83	40′
30′	.1495	6.69	30′
40′	.1524	6.56	20′
50′	.1554	6.43	10′
9°	.1584	6.31	**81°**
10′	.1614	6.197	50′
20′	.1644	6.084	40′
30′	.1673	5.976	30′
40′	.1703	5.871	20′
50′	.1733	5.769	10′
10°	.1763	5.671	**80°**
10′	.1793	5.576	50′
20′	.1823	5.485	40′
30′	.1853	5.396	30′
40′	.1883	5.309	20′
50′	.1914	5.226	10′
11°	.1944	5.145	**79°**
10′	.1974	5.066	50′
20′	.2004	4.989	40′
30′	.2035	4.915	30′
40′	.2065	4.843	20′
50′	.2095	4.773	10′
12°	.2126	4.705	**78°**
10′	.2156	4.638	50′
20′	.2186	4.574	40′
30′	.2217	4.511	30′
40′	.2247	4.449	20′
50′	.2278	4.390	10′
13°	.2309	4.331	**77°**
10′	.2339	4.275	50′
20′	.2370	4.219	40′
30′	.2401	4.165	30′
40′	.2432	4.113	20′
50′	.2462	4.061	10′
14°	.2493	4.011	**76°**
10′	.2524	3.962	50′
20′	.2555	3.914	40′
30′	.2586	3.867	30′
40′	.2617	3.821	20′
50′	.2648	3.776	10′
15°	.2679	3.732	**75°**
	ctn θ	tan θ	θ

φ	tan φ	ctn φ	
15°	.2679	3.732	**75°**
10′	.2711	3.689	50′
20′	.2742	3.647	40′
30′	.2773	3.606	30′
40′	.2805	3.566	20′
50′	.2836	3.526	10′
16°	.2867	3.487	**74°**
10′	.2899	3.450	50′
20′	.2931	3.412	40′
30′	.2962	3.376	30′
40′	.2994	3.340	20′
50′	.3026	3.305	10′
17°	.3057	3.271	**73°**
10′	.3089	3.237	50′
20′	.3121	3.204	40′
30′	.3153	3.172	30′
40′	.3185	3.140	20′
50′	.3217	3.108	10′
18°	.3249	3.078	**72°**
10′	.3281	3.047	50′
20′	.3314	3.018	40′
30′	.3346	2.989	30′
40′	.3378	2.960	20′
50′	.3411	2.932	10′
19°	.3443	2.904	**71°**
10′	.3476	2.877	50′
20′	.3508	2.850	40′
30′	.3541	2.824	30′
40′	.3574	2.798	20′
50′	.3607	2.773	10′
20°	.3640	2.747	**70°**
10′	.3673	2.723	50′
20′	.3706	2.699	40′
30′	.3739	2.675	30′
40′	.3772	2.651	20′
50′	.3805	2.628	10′
21°	.3839	2.605	**69°**
10′	.3872	2.583	50′
20′	.3906	2.560	40′
30′	.3939	2.539	30′
40′	.3973	2.517	20′
50′	.4006	2.496	10′
22°	.4040	2.475	**68°**
10′	.4074	2.455	50′
20′	.4108	2.434	40′
30′	.4142	2.414	30′
			67°
	ctn θ	tan θ	θ

Natural Tangents and Cotangents.

φ	tan φ	ctn φ		φ	tan φ	ctn φ		φ	tan φ	ctn φ	
22°				30°	.5774	1.732	60°	37°			
30′	.4142	2.414	30′	10′	.5812	1.720	50′	30′	.7673	1.303	30′
40′	.4176	2.394	20′	20′	.5851	1.709	40′	40′	.7720	1.295	20′
50′	.4210	2.375	10′	30′	.5890	1.698	30′	50′	.7766	1.288	10′
23°	.4245	2.356	67°	40′	.5930	1.686	20′	38°	.7813	1.280	52°
10′	.4279	2.337	50′	50′	.5969	1.675	10′	10′	.7860	1.272	50′
20′	.4314	2.318	40′	31°	.6009	1.664	59°	20′	.7907	1.265	40′
30′	.4348	2.300	30′	10′	.6048	1.653	50′	30′	.7954	1.257	30′
40′	.4383	2.282	20′	20′	.6088	1.643	40′	40′	.8002	1.250	20′
50′	.4417	2.264	10′	30′	.6128	1.632	30′	50′	.8050	1.242	10′
24°	.4452	2.246	66°	40′	.6168	1.621	20′	39°	.8098	1.235	51°
10′	.4487	2.229	50′	50′	.6208	1.611	10′	10′	.8146	1.228	50′
20′	.4522	2.211	40′	32°	.6249	1.600	58°	20′	.8195	1.220	40′
30′	.4557	2.194	30′	10′	.6289	1.590	50′	30′	.8243	1.213	30′
40′	.4592	2.177	20′	20′	.6330	1.580	40′	40′	.8292	1.206	20′
50′	.4628	2.161	10′	30′	.6371	1.570	30′	50′	.8342	1.199	10′
25°	.4663	2.145	65°	40′	.6412	1.560	20′	40°	.8391	1.192	50°
10′	.4699	2.128	50′	50′	.6453	1.550	10′	10′	.8441	1.185	50′
20′	.4734	2.112	40′	33°	.6494	1.540	57°	20′	.8491	1.178	40′
30′	.4770	2.097	30′	10′	.6536	1.530	50′	30′	.8541	1.171	30′
40′	.4806	2.081	20′	20′	.6577	1.520	40′	40′	.8591	1.164	20′
50′	.4841	2.066	10′	30′	.6619	1.511	30′	50′	.8642	1.157	10′
26°	.4877	2.050	64°	40′	.6661	1.501	20′	41°	.8693	1.150	49°
10′	.4913	2.035	50′	50′	.6703	1.492	10′	10′	.8744	1.144	50′
20′	.4950	2.020	40′	34°	.6745	1.483	56°	20′	.8796	1.137	40′
30′	.4986	2.006	30′	10′	.6787	1.473	50′	30′	.8847	1.130	30′
40′	.5022	1.991	20′	20′	.6830	1.464	40′	40′	.8899	1.124	20′
50′	.5059	1.977	10′	30′	.6873	1.455	30′	50′	.8952	1.117	10′
27°	.5095	1.963	63°	40′	.6916	1.446	20′	42°	.9004	1.111	48°
10′	.5132	1.949	50′	50′	.6959	1.437	10′	10′	.9057	1.104	50′
20′	.5169	1.935	40′	35°	.7002	1.428	55°	20′	.9110	1.098	40′
30′	.5206	1.921	30′	10′	.7046	1.419	50′	30′	.9163	1.091	30′
40′	.5243	1.907	20′	20′	.7089	1.411	40′	40′	.9217	1.085	20′
50′	.5280	1.894	10′	30′	.7133	1.402	30′	50′	.9271	1.079	10′
28°	.5317	1.881	62°	40′	.7177	1.393	20′	43°	.9325	1.072	47°
10′	.5354	1.868	50′	50′	.7221	1.385	10′	10′	.9380	1.066	50′
20′	.5392	1.855	40′	36°	.7265	1.376	54°	20′	.9435	1.060	40′
30′	.5430	1.842	30′	10′	.7310	1.368	50′	30′	.9490	1.054	30′
40′	.5467	1.829	20′	20′	.7355	1.360	40′	40′	.9545	1.048	20′
50′	.5505	1.816	10′	30′	.7400	1.351	30′	50′	.9601	1.042	10′
29°	.5543	1.804	61°	40′	.7445	1.343	20′	44°	.9657	1.036	46°
10′	.5581	1.792	50′	50′	.7490	1.335	10′	10′	.9713	1.030	50′
20′	.5619	1.780	40′	37°	.7536	1.327	53°	20′	.9770	1.024	40′
30′	.5658	1.767	30′	10′	.7581	1.319	50′	30′	.9827	1.018	30′
40′	.5696	1.756	20′	20′	.7627	1.311	40′	40′	.9884	1.012	20′
50′	.5735	1.744	10′	30′	.7673	1.303	30′	50′	.9942	1.006	10′
30°	.5774	1.732	60°				52°	45°	1.0000	1.000	45°
	ctn θ	tan θ	θ		ctn θ	tan θ	θ		ctn θ	tan θ	θ

Natural Secants and Cosecants

φ	sec φ	csc φ	
0°	1.0000		90°
10′	1.0000		50′
20′	1.0000		40′
30′	1.0000		30′
40′	1.0001		20′
50′	1.0001		10′
1°	1.0002	57.	89°
10′	1.0002	49.	50′
20′	1.0003	43.	40′
30′	1.0003	38.	30′
40′	1.0004	34.	20′
50′	1.0005	31.	10′
2°	1.0006	28.7	88°
10′	1.0007	26.5	50′
20′	1.0008	24.6	40′
30′	1.0010	22.9	30′
40′	1.0011	21.5	20′
50′	1.0012	20.2	10′
3°	1.0014	19.1	87°
10′	1.0015	18.1	50′
20′	1.0017	17.2	40′
30′	1.0019	16.4	30′
40′	1.0021	15.6	20′
50′	1.0022	15.0	10′
4°	1.0024	14.3	86°
10′	1.0026	13.76	50′
20′	1.0029	13.23	40′
30′	1.0031	12.75	30′
40′	1.0033	12.29	20′
50′	1.0036	11.87	10′
5°	1.0038	11.47	85°
10′	1.0041	11.10	50′
20′	1.0043	10.76	40′
30′	1.0046	10.43	30′
40′	1.0049	10.13	20′
50′	1.0052	9.84	10′
6°	1.0055	9.57	84°
10′	1.0058	9.31	50′
20′	1.0061	9.07	40′
30′	1.0065	8.83	30′
40′	1.0068	8.61	20′
50′	1.0072	8.40	10′
7°	1.0075	8.21	83°
10′	1.0079	8.02	50′
20′	1.0082	7.83	40′
30′	1.0086	7.66	30′
			82°
	csc θ	sec θ	θ

φ	sec φ	csc φ	
7°			
30′	1.009	7.66	30′
40′	1.009	7.50	20′
50′	1.009	7.34	10′
8°	1.010	7.19	82°
10′	1.010	7.04	50′
20′	1.011	6.90	40′
30′	1.011	6.77	30′
40′	1.012	6.64	20′
50′	1.012	6.51	10′
9°	1.012	6.39	81°
10′	1.013	6.277	50′
20′	1.013	6.166	40′
30′	1.014	6.059	30′
40′	1.014	5.955	20′
50′	1.015	5.855	10′
10°	1.015	5.759	80°
10′	1.016	5.665	50′
20′	1.016	5.575	40′
30′	1.017	5.487	30′
40′	1.018	5.403	20′
50′	1.018	5.320	10′
11°	1.019	5.241	79°
10′	1.019	5.164	50′
20′	1.020	5.089	40′
30′	1.020	5.016	30′
40′	1.021	4.945	20′
50′	1.022	4.876	10′
12°	1.022	4.810	78°
10′	1.023	4.745	50′
20′	1.024	4.682	40′
30′	1.024	4.620	30′
40′	1.025	4.560	20′
50′	1.026	4.502	10′
13°	1.026	4.445	77°
10′	1.027	4.390	50′
20′	1.028	4.336	40′
30′	1.028	4.284	30′
40′	1.029	4.232	20′
50′	1.030	4.182	10′
14°	1.031	4.134	76°
10′	1.031	4.086	50′
20′	1.032	4.039	40′
30′	1.033	3.994	30′
40′	1.034	3.950	20′
50′	1.034	3.906	10′
15°	1.035	3.864	75°
	csc θ	sec θ	θ

φ	sec φ	csc φ	
15°	1.035	3.864	75°
10′	1.036	3.822	50′
20′	1.037	3.782	40′
30′	1.038	3.742	30′
40′	1.039	3.703	20′
50′	1.039	3.665	10′
16°	1.040	3.628	74°
10′	1.041	3.592	50′
20′	1.042	3.556	40′
30′	1.043	3.521	30′
40′	1.044	3.487	20′
50′	1.045	3.453	10′
17°	1.046	3.420	73°
10′	1.047	3.388	50′
20′	1.048	3.356	40′
30′	1.049	3.326	30′
40′	1.049	3.295	20′
50′	1.050	3.265	10′
18°	1.051	3.236	72°
10′	1.052	3.207	50′
20′	1.053	3.179	40′
30′	1.054	3.152	30′
40′	1.056	3.124	20′
50′	1.057	3.098	10′
19°	1.058	3.072	71°
10′	1.059	3.046	50′
20′	1.060	3.021	40′
30′	1.061	2.996	30′
40′	1.062	2.971	20′
50′	1.063	2.947	10′
20°	1.064	2.924	70°
10′	1.065	2.901	50′
20′	1.066	2.878	40′
30′	1.068	2.855	30′
40′	1.069	2.833	20′
50′	1.070	2.812	10′
21°	1.071	2.790	69°
10′	1.072	2.769	50′
20′	1.074	2.749	40′
30′	1.075	2.729	30′
40′	1.076	2.709	20′
50′	1.077	2.689	10′
22°	1.079	2.669	68°
10′	1.080	2.650	50′
20′	1.081	2.632	40′
30′	1.082	2.613	30′
			67°
	csc θ	sec θ	θ

Natural Secants and Cosecants.

φ	sec φ	csc φ	
22°			
30′	1.082	2.613	30′
40′	1.084	2.595	20′
50′	1.085	2.577	10′
23°	1.086	2.559	**67°**
10′	1.088	2.542	50′
20′	1.089	2.525	40′
30′	1.090	2.508	30′
40′	1.092	2.491	20′
50′	1.093	2.475	10′
24°	1.095	2.459	**66°**
10′	1.096	2.443	50′
20′	1.097	2.427	40′
30′	1.099	2.411	30′
40′	1.100	2.396	20′
50′	1.102	2.381	10′
25°	1.103	2.366	**65°**
10′	1.105	2.352	50′
20′	1.106	2.337	40′
30′	1.108	2.323	30′
40′	1.109	2.309	20′
50′	1.111	2.295	10′
26°	1.113	2.281	**64°**
10′	1.114	2.268	50′
20′	1.116	2.254	40′
30′	1.117	2.241	30′
40′	1.119	2.228	20′
50′	1.121	2.215	10′
27°	1.122	2.203	**63°**
10′	1.124	2.190	50′
20′	1.126	2.178	40′
30′	1.127	2.166	30′
40′	1.129	2.154	20′
50′	1.131	2.142	10′
28°	1.133	2.130	**62°**
10′	1.134	2.118	50′
20′	1.136	2.107	40′
30′	1.138	2.096	30′
40′	1.140	2.085	20′
50′	1.142	2.074	10′
29°	1.143	2.063	**61°**
10′	1.145	2.052	50′
20′	1.147	2.041	40′
30′	1.149	2.031	30′
40′	1.151	2.020	20′
50′	1.153	2.010	10′
30°	1.155	2.000	**60°**
	csc θ	sec θ	θ

φ	sec φ	csc φ	
30°	1.155	2.000	**60°**
10′	1.157	1.990	50′
20′	1.159	1.980	40′
30′	1.161	1.970	30′
40′	1.163	1.961	20′
50′	1.165	1.951	10′
31°	1.167	1.942	**59°**
10′	1.169	1.932	50′
20′	1.171	1.923	40′
30′	1.173	1.914	30′
40′	1.175	1.905	20′
50′	1.177	1.896	10′
32°	1.179	1.887	**58°**
10′	1.181	1.878	50′
20′	1.184	1.870	40′
30′	1.186	1.861	30′
40′	1.188	1.853	20′
50′	1.190	1.844	10′
33°	1.192	1.836	**57°**
10′	1.195	1.828	50′
20′	1.197	1.820	40′
30′	1.199	1.812	30′
40′	1.202	1.804	20′
50′	1.204	1.796	10′
34°	1.206	1.788	**56°**
10′	1.209	1.781	50′
20′	1.211	1.773	40′
30′	1.213	1.766	30′
40′	1.216	1.758	20′
50′	1.218	1.751	10′
35°	1.221	1.743	**55°**
10′	1.223	1.736	50′
20′	1.226	1.729	40′
30′	1.228	1.722	30′
40′	1.231	1.715	20′
50′	1.233	1.708	10′
36°	1.236	1.701	**54°**
10′	1.239	1.695	50′
20′	1.241	1.688	40′
30′	1.244	1.681	30′
40′	1.247	1.675	20′
50′	1.249	1.668	10′
37°	1.252	1.662	**53°**
10′	1.255	1.655	50′
20′	1.258	1.649	40′
30′	1.260	1.643	30′
			52°
	csc θ	sec θ	θ

φ	sec φ	csc φ	
37°			
30′	1.260	1.643	30′
40′	1.263	1.636	20′
50′	1.266	1.630	10′
38°	1.269	1.624	**52°**
10′	1.272	1.618	50′
20′	1.275	1.612	40′
30′	1.278	1.606	30′
40′	1.281	1.601	20′
50′	1.284	1.595	10′
39°	1.287	1.589	**51°**
10′	1.290	1.583	50′
20′	1.293	1.578	40′
30′	1.296	1.572	30′
40′	1.299	1.567	20′
50′	1.302	1.561	10′
40°	1.305	1.556	**50°**
10′	1.309	1.550	50′
20′	1.312	1.545	40′
30′	1.315	1.540	30′
40′	1.318	1.535	20′
50′	1.322	1.529	10′
41°	1.325	1.524	**49°**
10′	1.328	1.519	50′
20′	1.332	1.514	40′
30′	1.335	1.509	30′
40′	1.339	1.504	20′
50′	1.342	1.499	10′
42°	1.346	1.494	**48°**
10′	1.349	1.490	50′
20′	1.353	1.485	40′
30′	1.356	1.480	30′
40′	1.360	1.476	20′
50′	1.364	1.471	10′
43°	1.367	1.466	**47°**
10′	1.371	1.462	50′
20′	1.375	1.457	40′
30′	1.379	1.453	30′
40′	1.382	1.448	20′
50′	1.386	1.444	10′
44°	1.390	1.440	**46°**
10′	1.394	1.435	50′
20′	1.398	1.431	40′
30′	1.402	1.427	30′
40′	1.406	1.423	20′
50′	1.410	1.418	10′
45°	1.414	1.414	**45°**
	csc θ	sec θ	θ

EXPLANATION OF THE TABLES.

§ 1. TABLES IN GENERAL.

a. One quantity is said to be a **function** of another, when the former quantity is regarded as determined by the latter, according to some rule or formula. E. g. x^2, $\sqrt{x}$, $\log x$, $\sin x$, $\log \sin x$, are all called functions of x. **A mathematical table** is an orderly arrangement of the values of some function for certain selected values of the quantity by which it is regarded as determined. The successive values of the latter quantity are assumed arbitrarily, and generally at equal intervals; and this quantity is called the **argument** of the table. Some functions require several independent quantities for their determination; and the corresponding tables are tables of several arguments. Thus, a multiplication-table is a table of *two* arguments; namely, the two factors.

b. A table may be used in two ways: **directly** and **inversely**. The direct use of the table consists in finding the value of the function for an assumed value of the argument; the inverse use, in finding the value of the argument for an assumed value of the function.

c. Before beginning to use any table, the student should give attentive consideration to its arrangement, and to the best mode of employing it with accuracy and ease. Every feature of it should be carefully examined, and the explanations which are attached to it should be fully mastered. The time thus spent will be time gained, contributing not only to power in computation, but also, very materially, to the thorough practical knowledge of the nature of the tabulated functions.

§ 2. INTERPOLATION.

a. **Interpolation** consists in finding the value of one of the two quantities, argument and function, for an assumed value of the other quantity, lying between two successive tabulated values. Most mathematical tables are so constructed as to admit of interpolation by the principle that *corresponding non-tabulated values of the function and argument lie between corresponding tabulated values and divide the differences between them in the same ratio.* This is the principle of **proportional parts.** Let x_1 and x_2 be two successive tabulated values of the argument of a table, and u_1 and u_2 the corresponding values of the function. Then, $x_2 - x_1$ and $u_2 - u_1$ are called corresponding tabular differences. We shall denote these differences by Δx and Δu. If, now, x and u are corresponding values of the function and argument, of which one is known to lie between the two above-cited tabulated values of the same quantity, the principle of proportional parts is that if

$$\lambda = \frac{x - x_1}{\Delta x}, \qquad \lambda' = \frac{x_2 - x}{\Delta x} = 1 - \lambda,$$

$$\mu = \frac{u - u_1}{\Delta u}, \qquad \mu' = \frac{u_2 - u}{\Delta u} = 1 - \mu,$$

then (to the limit of accuracy belonging to the table)

$$\lambda = \mu, \qquad \lambda' = \mu',$$

or,

$$u = u_1 + \lambda \Delta u = u_2 - \lambda' \Delta u,$$
$$x = x_1 + \mu \Delta x = x_2 - \mu' \Delta x.$$

Thus, the required value of the function or argument may be obtained by applying a **correction** to either of the two tabulated values between which the required value lies. In computing this correction, the *signs* of the differences employed must be carefully observed. If x_1 and x_2 are so chosen as to make Δx positive, Δu may be either positive or negative. In the former case, the function is said to be *increasing*; in the latter, *decreasing*.

b. Either of the two formulas given above for finding u may be employed, in interpolation, in the direct use of the table; either of the formulas for x may be employed in the inverse use of the table. In most tables, $\Delta x =$ *one unit* in the last numeral place of the tabulated values of x. Hence λ is composed of *the figures which follow that numeral place* in the given non-tabulated value of the argument, preceded by a decimal-point; while λ' is the *complement* of λ (that is, can be found by subtracting from 9 each figure of λ except the last, and subtracting that from 10). The correction for u is, therefore, found simply by *multiplying* the figures in question into Δu, and pointing off according to the case; x will be corrected by *annexing* to x_1 the figures of $\frac{u - u_1}{\Delta u}$, or the figures complementary to $\frac{u_2 - u}{\Delta u}$.

c. In some of the tables of this collection will be found, set against each value of the function, *a number in small type*, which shows what Δu would be if the function varied through a whole interval corresponding to Δx at the same rate at which it is changing when it passes through the value against which this number is set. This number may be called the *rate of difference*, or simply the *difference*, of u, and may be substituted for Δu in the formulas of interpolation. But, in that case, we ought to work from the NEAREST tabulated value of x or u; that is, from x_1 or u_1 when λ or $\mu < 0.5$, and from x_2 or u_2 when λ' or $\mu' < 0.5$. (See examples in the explanation of the table of *Logarithms of Circular Functions.*)

d. An interpolated value of the function should not be carried out beyond the last numeral place of the tabulated value from which it is computed; so that, in finding $\lambda \Delta u$ or $\lambda' \Delta u$, we should reject the decimal part of the product, Δu being regarded as an integer. Owing to the combination of the figures rejected in the correction and those omitted in the tabulated value of the function, an interpolated value is liable to an error of ±1 in the last figure.

The number of figures annexed to the tabulated value of the argument, in inverse interpolation, should be less by one than the number of figures contained in Δu*.* It is sometimes, indeed, made *equal* to the latter number (and will always be, if Δu consists of only one figure); but, in that case, the *last figure* must be regarded as *uncertain.* When the given value of the function is the result of computation, of course this uncertainty may extend back to earlier figures.

e. In taking the correction of either the function or the argument only to a certain number of figures, we must observe the following rule, which is a universal rule of computation:—

Whenever figures are neglected at the end of a number, if the figures neglected amount to more than half a unit in the place of the last figure retained, the last figure retained must be increased by 1. E. g. 27.528 = 27.53 to the nearest hundredth = 27.5 to the nearest tenth = 28 to the nearest unit = 30 to the nearest ten.

f. The various rules of interpolation will be found to be fully exemplified below, in the explanations of the tables of *Logarithms* and *Logarithms of Circular Functions.*

g. In interpolating in some tables (e. g. in VLACQ's great ten-place table), we must have regard to **second differences**, or differences between differences. In this case, we add to the above formulas for u the term

$$-\tfrac{1}{2}\lambda\lambda'\Delta^2 u,$$

where $\Delta^2 u$ denotes the second difference of u, taken positively when Δu is increasing. The greatest value of this term is *one eighth* of $\Delta^2 u$, so that it is insignificant when $\Delta^2 u < 4$. *In the present tables this term may always be neglected;* although it is useful as measuring the extent of error, and may occasionally guide the judgment of the computer when the fractional part of the correction is equal, or nearly equal, to 0.5. But where such nicety of work seems to be called for, it is best to use a table of a larger number of places.

§ 3. PROPORTIONAL PARTS.

a. The table of **Proportional Parts** (folded page) may be used in connexion with any other table, as an aid in *interpolation.* It contains the product of every integer from 1 to 100 by every *tenth* from 0.1 to 0.9. If the multiplier consists of one figure in any other numeral place, it is only necessary to change the position of the decimal-point in the product. To multiply a number of two figures by any decimal whatever, we must find the products which correspond to the successive figures of the multiplier, and add them together. The decimal part of the result is generally to be discarded, and in that case the general rule given above (in § 2, *e*) must be observed. Thus, let it be required to find 0.619 × 37. Looking in the column belonging to 37, we find

$$\begin{aligned} 0.6 \times 37 &= 22.2 \\ 0.01 \times 37 &= 0.37 \\ 0.009 \times 37 &= \underline{0.333} \\ \therefore\ 0.619 \times 37 &= 23. \end{aligned}$$

In like manner, we find

$0.27 \times 15 = 4,\qquad 0.59 \times 73 = 43,\qquad 0.78 \times 69 = 54,$
$0.96 \times 84 = 81,\qquad 0.36 \times 57 = 21,\qquad 0.289 \times 51 = 15,$
$0.483 \times 93 = 45,\qquad 0.374 \times 82 = 31,\qquad 0.053 \times 68 = 4.$

b. This table can also be used *inversely*. Thus, let it be required to find, to two decimal-places, what part 36 is of 79. Looking in the column of 79, we find

$$\begin{array}{rcl} & & 36 \\ 0.4 \times 79 & = & \underline{31.6} \\ & & 4.4 \end{array}$$

$$0.06 \times 79 = 4.74 \text{ (the \textit{nearest} product)}$$

$$\therefore \frac{36}{79} = 0.46.$$

In like manner, we find

$$\frac{29}{68} = 0.43,\quad \frac{72}{89} = 0.81,\quad \frac{31}{98} = 0.32,\quad \frac{26}{71} = 0.37,\quad \frac{45}{57} = 0.79,\quad \frac{11}{37} = 0.30.$$

A little practice will enable the student to use this table easily and rapidly.

§ 4. LOGARITHMS.

a. **Denary**, or **Briggsian**, logarithms, being those employed in actual computation, are always referred to, in this collection of tables, when the term *logarithm* is used without qualification. The **characteristic**, or integral part, of the denary logarithm of a number depends only on the position of the first significant figure of the number relatively to the units' place, and may be found by a well-known rule; the **mantissa**, or fractional part, depends only on the series of significant figures which compose the number, and is the only part of the logarithm for which it is necessary to employ a table. A table of logarithms is complete, to an assigned number of places, if it gives (explicitly or by interpolation) to that number of places the mantissa of the logarithm of every possible series of significant figures. Denary logarithms are, in general, incommensurable numbers, and cannot, therefore, be exactly expressed in figures. They are variously given, in different tables, to ten, seven, six, five, four, and three places of decimals. Four-place logarithms are sufficient for the ordinary purposes of engineering, navigation, the work of the physical and chemical laboratory, and many of the subordinate computations of astronomy; and, in most of these cases, are all that the accuracy of the data will justify us in using. Seven places are, however, needful for the more accurate kinds of astronomical and geodetic work.

b. If one number is the logarithm of another, the second number is called the **antilogarithm** of the first. This relation is denoted by the symbol $\log^{-1}$. Thus, if $u = \log x$, then $x = \log^{-1} u$. In an ordinary table of logarithms, the *argument* is the antilogarithm, which is tabulated to a greater or less number of figures, according to the number of places to which the logarithm is given, and the *function* is the mantissa of the logarithm, which we often speak of simply as the *logarithm*.

To find the logarithm of any number.

c. If the number consists of *three significant figures*, seek the first two significant figures in the first column of the table of **Logarithms** (pp. 2, 3), and the third at the top of the table. In the line and column thus determined will be found the mantissa of the required logarithm, printed without the decimal-point. Find the characteristic by the rule, and prefix it, with the decimal-point, to the mantissa. E. g., log 2870 = 3.4579. If the given number has *less than three* significant figures, fill it out to three figures by anexing a zero or zeros. E. g., log 0.35 = log 0.350 = 9.5441 — 10, log 6 = log 6.00 = 0.7782. If the number has *more than three* significant figures, its logarithm must be found by one of the formulas of interpolation given above. The rule is: — *Find the logarithm of the first three significant figures of the given number and also that of the next following number of three figures* (1000 following 999); *then apply to* EITHER *of these two logarithms a correction, obtained by multiplying the difference between them by the difference between the given number and the three-figure number which corresponds to the logarithm chosen to be corrected, and rejecting* (*with due attention to the rule of* § 2, *e*) *as many figures at the end of the product as are contained in the latter difference.* The table of **Proportional Parts** may be employed in performing the multiplications. Thus, to find log 5668.4. Using the notation of the formulas of interpolation, and remembering that the place of the decimal-point in the given number may be disregarded in finding the mantissa of the required logarithm, we have

$$\begin{aligned} x_1 &= 566, & u_1 &= \text{mant} \log x_1 = 7528, \\ x_2 &= 567, & u_2 &= \text{mant} \log x_2 = 7536, \\ \Delta x &= 1, & \Delta u &= \phantom{\text{mant} \log x_2 = 753}8; \end{aligned}$$

so that log x may be found by either of the following methods: —

$$\lambda = 0.84, \qquad \lambda \Delta u = 6.72 = 7 \text{ to units}, \qquad u = 7528 + 7 = 7535;$$

or,

$$\lambda' = 1 - \lambda = 0.16, \qquad \lambda' \Delta u = 1.28 = 1 \text{ to units}, \qquad u = 7536 - 1 = 7535.$$

$$\therefore \log 5668.4 = 3.7535.$$

Let the beginner find the following logarithms by this method: —

log 59.43 = 1.7740,
log 284.8 = 2.4545,
log 0.073748 = 8.8678 — 10,
log 3.1607 = 0.4998,
log 0.0081472 = 7.9110 — 10,
log 572820 = 5.7581,
log 0.50167 = 9.7004 — 10,
log 99968. = 4.9999.

The interpolated logarithm should never be carried to more than four decimal-places.

d. The work of interpolation may be shortened by using the column of proportional parts, marked P. P., on the right of the table. In using this column, one must work from the three-figure number NEAREST to the given number (in the above example, from 567, not from 566). If the given number has only four figures, so that λ or λ' has only one figure, then the correction will be found in the column P. P., under λ or λ' (according as we are working from the number below or the number above the given number), and in the same line with the logarithm to be corrected. If the given number has more than four figures, the correction must be estimated by the observation of the corrections which correspond to the figures below and above the first figure of λ or λ'. E. g.

mant log 2848 = mant log 285 — cor. for .2 = 4548 — 3 = 4545;
mant log 56684 = mant log 567 — cor. for .16 = 7536 — 2 = 7534.

In the last case the correction is either 1 or 2, and, since .16 is nearer .20 than .10, we choose the correction belonging to .20. Larger tables show that the mantissa of the required logarithm, to five places, is 75346; so that the value found by the column P. P. is here nearly as accurate as that obtained by computation. There is a slightly greater liability to error when we use the column P. P. than when we interpolate by computation; but the disadvantage is generally insignificant. The last figure of an interpolated logarithm obtained from any table may always be one unit in error. E. g.: the true mant log of 57282 to five places is 75802; and this is a case in which the column P. P. gives a better result than computation.

The student is advised now to find all the logarithms in the above list by using the column P. P.

e. If the first figure of the given number is 1, it will be found tabulated to *four* figures in pp. 4, 5. The correction for a fifth and following figures may be found by the method of interpolation explained in *c*. As the differences are always small on these pages, and the corrections easily computed, the column P. P. is not here given; but, to facilitate taking the last difference, we have printed at the end of each line, under the heading 10, the first logarithm of the following line. Let the student find the following logarithms: —

log 11.737 = 1.0696, log 0.00100066 = 7.0003 — 10,
log 0.15703 = 9.1960 — 10, log 18597. = 4.2694.

To find the antilogarithm of any logarithm.

f. It is enough to explain the way of finding the series of significant figures which compose the antilogarithm, by means of the mantissa of the given logarithm; the pointing off of the antilogarithm being determined, according to rule, by the given characteristic. If the mantissa of the given logarithm is contained in the table, the required antilogarithm is at once found by inspection. Otherwise, we must resort to the formulas of interpolation, which give the following rule: — *Find two successive tabulated logarithms* (u_1 and u_2) *between which the given logarithm* (u) *lies; then divide the difference between either of these tabulated logarithms and the given logarithm* ($u - u_1$ or $u_2 - u$) *by the difference between the tabulated logarithms* (Δu), *carry out the quotient to the* NEAREST *tenth* (that is, to one figure, which may be 0), *and add it to or subtract it from the antilogarithm* (x_1 or x_2) *of the tabulated logarithm* (u_1 or u_2) *with which the given logarithm has been compared.* The antilogarithm is always a figure *annexed* to the three or four tabulated figures of x_1.

The division should not generally be carried beyond one figure. Even the first figure is, in most cases, somewhat uncertain. If the mantissa of the given logarithm is less than 3010, pp. 4, 5 should be used. On pp. 2, 3, the column P. P. may be employed.

Let it be required to find $\log^{-1} 1.5284$. We find

$$\begin{array}{llll} u_1 = 5276, & x_1 = 337, & u - u_1 = 8, & u_2 - u = 5, \\ u_2 = 5289, & x_2 = 338, & \mu = \mu \Delta x = \dfrac{8}{13} = 0.6\,.\,., & \\ \Delta u = \ \ 13, & \Delta x = \ \ 1, & \mu' = \mu' \Delta x = \dfrac{5}{13} = 0.4\,.\,., & \end{array}$$

$$x = 337 + 0.6 = 338 - 0.4 = 337.6;$$

$$\therefore \log^{-1} 1.5284 = 33.76.$$

More briefly, looking along the line of $u_2 = 5289$ for $5 = u_2 - u$ in column P. P., we find that 5 corresponds to the correction 4, which gives at once the required number. In like manner, the student may find

$$\begin{array}{ll} \log^{-1} 1.9155 = 82.32, & \log^{-1} (5.8760 - 10) = 0.00007517, \\ \log^{-1} 3.8291 = 6747, & \log^{-1} (9.5727 - 10) = 0.3738, \\ \log^{-1} 0.1548 = 1.4283, & \log^{-1} (8.2731 - 10) = 0.018755. \end{array}$$

g. The convenient usage of *making negative characteristics positive,* by the addition of 10, is followed, throughout the present collection of tables, whenever logarithms are printed with their characteristics. This must be always understood, though no explicit reference be made to it in the explanation of the table.

§ 5. LOGARITHMS OF SUMS AND DIFFERENCES.

a. This is one form of a table devised by GAUSS to facilitate finding the logarithm of the *sum* or *difference* of two numbers which are themselves given only by their logarithms. The *argument* of the table is any logarithm, and may be called $\log x$; the *function* tabulated is then $\log (x + 1)$. It follows that, if the function is denoted by $\log x$, the argument is $\log (x - 1)$. The function may be called the **Gaussian** of the argument, and the argument the **anti-Gaussian** of the function; and the symbols $\mathfrak{G}$ and $\mathfrak{G}^{-1}$ may be used to denote these relations. Thus we have

$$\log (x + 1) = \mathfrak{G} \log x, \qquad \log (x - 1) = \mathfrak{G}^{-1} \log x.$$

b. **To find the Gaussian of a given logarithm.** Seek the characteristic of the given logarithm (increased by 10 if negative) at the top of the table, and the first two figures of the mantissa in the left-hand column. If the third and fourth figures of the mantissa are zero, the Gaussian will be found in the column and line thus determined; otherwise, it can be obtained by the method of interpolation which has been fully explained in § 2. In three columns of the table, the *rate of difference* of the Gaussian is printed in small type after the value of the function, and may be used instead of the tabular difference of the Gaussian *through half the tabular interval before and after* the value to which it is attached, as explained in

§ 2, *c*, and completely illustrated below, in the explanation of the table of *Logarithms of Circular Functions*. The table of Proportional Parts may be employed in computing the corrections. Examples: —

$$\mathfrak{G}\ 1.0960 = 1.1295, \qquad \mathfrak{G}\ (7.5265 - 10) = 0.0015,$$
$$\mathfrak{G}\ 3.8129 = 3.8130, \qquad \mathfrak{G}\ (9.6431 - 10) = 0.1582.$$

If $\log x < 6.0000 - 10$, $\log (x + 1) = 0.0000$ to four places;
if $\log x > 4.0000$, $\log (x + 1) = \log x$ to four places.

c. **To find the anti-Gaussian of a given logarithm.** Seek, in the body of the table, two successive logarithms between which the given logarithm lies, and then find the corresponding value of the argument by interpolation. Examples: —

$$\mathfrak{G}^{-1}\ 1.0960 = 1.0597, \qquad \mathfrak{G}^{-1}\ 0.1051 = 9.4373 - 10,$$
$$\mathfrak{G}^{-1}\ 3.8129 = 3.8128, \qquad \mathfrak{G}^{-1}\ 1.0216 = 0.9782.$$

d. **To find the logarithm of the sum or difference of the antilogarithms of two given logarithms.** If m and n are two numbers,

$$m + n = n\left(\frac{m}{n} + 1\right), \qquad m - n = n\left(\frac{m}{n} - 1\right),$$

$$\log (m + n) = \log n + \log\left(\frac{m}{n} + 1\right) = \log n + \mathfrak{G} \log \frac{m}{n},$$

$$\log (m - n) = \log n + \log\left(\frac{m}{n} - 1\right) = \log n + \mathfrak{G}^{-1} \log \frac{m}{n}.$$

Example: —

Given $a = 4.142$, $b = 2.399$; to find $\sqrt{(a^2 + b^2)}$ and $\sqrt{(a^2 - b^2)}$.

$$\begin{array}{llll}
\log a & = 0.6172, & \log b & = 0.3800, \\
\log a^2 & = 1.2344, & \log b^2 & = 0.7600, \\
\log \dfrac{a^2}{b^2} & = 0.4744; & & \\
\mathfrak{G} \log \dfrac{a^2}{b^2} & = 0.6000, & \mathfrak{G}^{-1} \log \dfrac{a^2}{b^2} & = 0.2970, \\
\log b^2 & = 0.7600, & \log b^2 & = 0.7600, \\
\log (a^2 + b^2) & = 1.3600, & \log (a^2 - b^2) & = 1.0570, \\
\log \sqrt{(a^2 + b^2)} & = 0.6800, & \log \sqrt{(a^2 - b^2)} & = 0.5285, \\
\sqrt{(a^2 + b^2)} & = 4.787; & \sqrt{(a^2 - b^2)} & = 3.377.
\end{array}$$

§ 6. CIRCULAR, OR TRIGONOMETRIC, FUNCTIONS: NATURAL VALUES.

a. Three tables of the natural values of the trigonometric functions are given on pp. 22–27. Each table is broken up into six divisions, and occupies two pages. The argument is the angle, which is tabulated at intervals of 10′ from 0° to 90°. Angles in the *first half of the quadrant* will be found in the *left-hand column* of the several divisions of the table, and for those angles the names of the functions are to be taken from the *top* of the page; angles in the *second half of the quadrant* are to be found in the *right-hand* *column* of the table, and for those angles the names of the functions are to be taken from the *bottom* of the page. The *angles* standing at the right and left in the same line are *complements* of each other; and the names of the *functions* at the top and bottom of the same column are *complementary*. The value of any of the functions for a non-tabulated angle, or the value of the angle for a non-tabulated value of one of the functions, can be found by the method of interpolation explained in § 2. The precepts of § 2, *d*, *e*, should be observed in computing the corrections. The tabulated values of the functions are generally given to four significant figures; but, in the tables of tangents and secants, they are sometimes given to a less number of figures (to avoid errors in interpolation), and are sometimes omitted altogether. In these cases the functions can be best found by finding their logarithms by the table of *Logarithms of Circular Functions* (see § 7), and then the numbers corresponding by the table of *Logarithms*.

b. To find any function of an angle *greater than* 90°, we must subtract from the given angle the greatest multiple of 90° which it contains; if an *even* multiple has been subtracted, we look out the required function of the remainder; if an *odd* multiple, the complementary function; and we then fix the *sign* of the function by considering the *quadrant* in which the given angle lies. For a *negative* angle, we find the required function of the corresponding positive angle, and then fix its sign by considering the quadrant of the angle.

c. Examples of the use of these tables:—

$$\begin{array}{lll}
\sin 77^\circ 37' = 0.9767, & \tan 53^\circ 04' = 1.330, & \sec 68^\circ 45' = 2.759,\\
\cos 16^\circ 19' = 0.9597, & \mathrm{ctn}\ 3^\circ 18' = 17.4, & \csc 55^\circ 13' = 1.217;
\end{array}$$

$$\begin{array}{lll}
\sin 257^\circ 37' = -0.9767, & \tan 93^\circ 18' = -17.4, & \sec 325^\circ 13' = 1.217,\\
\cos 163^\circ 41' = -0.9597, & \mathrm{ctn}\ 323^\circ 04' = -1.330, & \csc 158^\circ 45' = 2.759;
\end{array}$$

$$\begin{array}{lll}
\sin(-257^\circ 37') = 0.9767, & \tan(-93^\circ 18') = 17.4, & \sec(-325^\circ 13') = 1.217,\\
\cos(-163^\circ 41') = -0.9597, & \mathrm{ctn}(-323^\circ 04') = 1.330, & \csc(-158^\circ 45') = -2.759;
\end{array}$$

$$\begin{array}{llll}
\sin^{-1} 0.2000 & = 11^\circ 32' \text{ or } = & 168^\circ 28' \text{ or } = & 371^\circ 32', \text{ etc.},\\
\cos^{-1}(-0.3542) & = 110^\circ 45' \text{ or } = & 249^\circ 15' \text{ or } = & 830^\circ 45', \text{ etc.},\\
\tan^{-1}(-4.570) & = 102^\circ 21' \text{ or } = & 282^\circ 21' \text{ or } = & -77^\circ 39', \text{ etc.},\\
\mathrm{ctn}^{-1}\ 0.3163 & = 72^\circ 27' \text{ or } = & 252^\circ 27' \text{ or } = & -107^\circ 33', \text{ etc.},\\
\sec^{-1} 5.000 & = 78^\circ 28' \text{ or } = & -78^\circ 28' \text{ or } = & \pm 281^\circ 32', \text{ etc.},\\
\csc^{-1}(-3.529) & = -16^\circ 28' \text{ or } = & 196^\circ 28' \text{ or } = & -163^\circ 32', \text{ etc.}
\end{array}$$

§ 7. LOGARITHMS OF CIRCULAR FUNCTIONS.

To find the logarithm of any circular function of a given angle.

a. If the angle is *less than* 6°, the part of the table which occupies the upper half of p. 10 may be used. (See also *g*.) The *left-hand division* of this part of the table gives the values of a logarithm S (the characteristic and the first two figures of the mantissa being printed at the head of the column), with the angular limits between which each value may be used. Thus, for all positive angles less than 1° 51′.479, S = 6.4637; for all angles between 1° 51′.479 and 2° 49′.567, S = 6.4636; etc. The *next following* *division* gives, in like manner, the values of a logarithm T. We must find *the logarithm of the angle*, reduced to minutes and decimals of a minute, and must then apply the formulas: —

$$\log \sin \phi = \log (\phi \text{ in minutes}) + \mathrm{S} - 10,$$
$$\log \tan \phi = \log (\phi \text{ in minutes}) + \mathrm{T} - 10.$$

The two right-hand divisions of this part of the table give the values of the log sec, with the angular limits for each value. The logarithms of the cosine, cotangent, and cosecant are the *arithmetical complements* (— 10) of the logarithms of the secant, tangent, and sine, respectively. Example: —

$$\text{l sin } 3^\circ\, 15'.23 = 8.7541, \quad \text{l tan } 3^\circ\, 15'.23 = 8.7548, \quad \text{l sec } 3^\circ\, 15'.23 = 0.0007,$$
$$\text{l csc } 3^\circ\, 15'.23 = 1.2459, \quad \text{l ctn } 3^\circ\, 15'.23 = 1.2452, \quad \text{l cos } 3^\circ\, 15'.23 = 9.9993;$$

the negative characteristics being here, as in the following examples, made positive by the addition of 10.

b. If the angle is *acute and greater than* 84°, we must take its *complement*, and then seek the *function complementary to that required*, for the angle thus obtained, by the method just expounded. Example: —

$$\text{l sin } 86^\circ\, 44'.77 = 9.9993, \quad \text{l tan } 86^\circ\, 44'.77 = 1.2452, \quad \text{l sec } 86^\circ\, 44'.77 = 1.2459,$$
$$\text{l csc } 86^\circ\, 44'.77 = 0.0007, \quad \text{l ctn } 86^\circ\, 44'.77 = 8.7548, \quad \text{l cos } 86^\circ\, 44'.77 = 8.7541.$$

c. If the angle is contained *between* 6° *and* 84°, we use the main part of the table, occupying the lower half of p. 10 and pp. 11–15. The angle is tabulated at intervals of 10′, from 6° to 45° in the left-hand column of the table, and from 45° to 84° in the right-hand column. The names of the functions are to be taken from the *tops* of the columns, when the angle is on the *left*; and from the *bottoms* of the columns, when the angle is on the *right*. The angles on the right and left of any line and the names at the top and bottom of any column have the same relation to each other as in the tables of *Natural Values* (§ 6). The true characteristic in the first, third, and sixth columns is —1, but is printed 9. The six columns are arranged in pairs. The two functions in each pair of columns are *reciprocal* to each other; and the logarithms are therefore *complementary*, and their differences are equal in value, with opposite signs. Down the middle of each double column are printed, in small type, the *rates of difference* of the logarithms in that double column. Each value of this rate may be used in interpolation, instead of Δ *u*, *through half the interval before and after the line on which it stands*, as stated in § 2, *c*. Thus, in finding the logarithms of the circular functions of any angle between 25° 25′ and 25° 35′ we work from the values corresponding to 25° 30′, the *nearest* tabulated angle; and compute the corrections by taking proportional parts of 26, 33, and 6, for the three pairs of functions. In applying the corrections, we must carefully observe, for each function, *whether the function is increasing or decreasing*.

For example, let the logarithms of the circular functions of 25° 27′.4 = 25° 30′ — 02′.6 be required. We find

$$\begin{array}{rlrlrl}
\text{l sin } 25^\circ\, 30' & = 9.6340 & \text{l tan } 25^\circ\, 30' & = 9.6785 & \text{l sec } 25^\circ\, 30' & = 0.0445 \\
0.26 \times 26 & = 7 & 0.26 \times 33 & = 9 & 0.26 \times 6 & = 2 \\
\text{l sin } 25^\circ\, 27'.4 & = \overline{9.6333}, & \text{l tan } 25^\circ\, 27'.4 & = \overline{9.6776}, & \text{l sec } 25^\circ\, 27'.4 & = \overline{0.0443},
\end{array}$$

l csc 25° 30′ = 0.3660	l ctn 25° 30′ = 0.3215	l cos 25° 30′ = 9.9555
7	9	2
l csc 25° 27′.4 = 0.3667,	l ctn 25° 27′.4 = 0.3224,	l cos 25° 27′.4 = 9.9557.

In like manner, we have

l sin 74° 46′ = 9.9845,	l tan 74° 46′ = 0 5649,	l sec 74° 46′ = 0.5804,
l csc 74° 46′ = 0.0155,	l ctn 74° 46′ = 9.4351,	l cos 74° 46′ = 9.4196.

d. If the angle is *greater than* 90°, or *negative*, we must use the method explained in § 6, *b*, for the tables of *Natural Values* of the circular functions. When the natural value of a circular function is negative, this should be indicated by writing the letter *n* after its logarithm. Examples:

l sin 105° 14′ = 9.9845, l tan 105° 14′ = 0.5649 *n*,
l sec 105° 14′ = 0.5804 *n*,

l csc 164° 46′ = 0.5804, l ctn 164° 46′ = 0.5649 *n*,
l cos 164° 46′ = 9.9845 *n*,

l sin (—74° 46′) = 9.9845 *n*, l tan (—105° 14′) = 0.5649,
l cos (—394° 46′) = 9.4196.

Given the logarithm of any circular function, to find the value of the corresponding angle.

e. If the given logarithm lies *without the limits of the main part of the table*, the upper part of p. 10 may be used. If the given logarithm is a *log sin less than* 9.0192, or a *log tan less than* 9.0216, subtract from it the proper value of S or T (or add the arithmetical complement), and the remainder is the log of the required angle in minutes. The limiting values of the log sin and log tan for each value of S and T are given in the table. If the given log is a *log csc greater than* 0.9808, or a *log ctn greater than* 0.9784, its arithmetical complement will be a log sin less than 9.0192, or a log tan less than 9.0216. If the given log is a *log sec less than* 0.0024, the limits between which the required angle lies are given by the table; the angle may have any value between these limits, and is not therefore very closely determined. If the given log is a *log cos greater than* 9.9976, its arithmetical complement is a log sec less than 0.0024.

If the given log is a *log sin, log tan, or log sec greater than* 9.9976, 0.9784, *or* 0.9808 (respectively), or a *log csc, log ctn, or log cos less than* 0.0024, 9.0216, *or* 9.0192 (respectively), we must change the name of the function to the *complementary name* (sin to cos, etc.), *find the corresponding angle* as above, and take the *complement* of the angle thus found. Examples: —

$(\text{log sin})^{-1}\ 8.9542 = 5^\circ\ 09'.8$, $(\text{log ctn})^{-1}\ 2.0531 = 0^\circ\ 30'.42$,
$(\text{log ctn})^{-1}\ 9.0024 = 84^\circ\ 15'.5$, $(\text{log sin})^{-1}\ 9.9983 = 84^\circ\ 56' \pm 4\tfrac{1}{2}'$.

f. If the given logarithm is contained *within the limits of the main part of the table*, the required angle is found by ordinary interpolation; and we may use the printed rate of difference as the value of Δu, working in each case from the *nearest* tabulated value. The angle should be found to the *nearest minute*, or, when the difference exceeds 100, to the nearest *tenth* of a minute. But in the right-hand pair of columns, the last figure of the angle thus found will generally be uncertain. Examples: —

Let it be required to find $(\log \sec)^{-1}\, 0.0643$; *i. e.* the angle of which the log sec is 0.0643. The nearest tabulated log sec is 0.0647. We have, then,

$$(\log \sec)^{-1}\, 0.0647 = 30^\circ\, 30', \; u_2 - u = 4, \; \Delta u = 7, \; \frac{4}{7} = 0.6,$$

$$\therefore \; (\log \sec)^{-1}\, 0.0643 = 30^\circ\, 30' - 06' = 30^\circ\, 24'.$$

In like manner, let the student find

$(\log \sin)^{-1}\, 9.5663 = 21^\circ\, 37'$,
$(\log \cos)^{-1}\, 9.9188 = 33^\circ\, 58'$ or $57'$,
$(\log \tan)^{-1}\, 0.7507 = 79^\circ\, 56'$,
$(\log \mathrm{ctn})^{-1}\, 0.0496 = 41^\circ\, 44'$,
$(\log \sec)^{-1}\, 0.2272 = 53^\circ\, 39'$,
$(\log \csc)^{-1}\, 0.1433 = 45^\circ\, 58'$.

The angle may also be found by the next following table.

g. Pp. 8 and 9 may also be used for angles less than 6° or greater than 84°.

E.g. $\mathrm{l}\,\sin 4^\circ\, 03'.4 = 8.8497$, $\mathrm{l}\,\tan 4^\circ\, 03'.4 = 8.8508$, $\mathrm{l}\,\sec 4^\circ\, 03'.4 = 0.0011$, $\mathrm{l}\,\csc 4^\circ\, 03'.4 = 1.1503$, $\mathrm{l}\,\mathrm{ctn}\, 4^\circ\, 03'.4 = 1.1492$, $\mathrm{l}\,\cos 4^\circ\, 03'.4 = 9.9989$.

§ 8. INVERSE CIRCULAR FUNCTIONS.

a. The table having this heading (pp. 16–18) is a table for finding the angle which corresponds to the given logarithm of a circular function. The logarithm (increased by 10) is the *argument* of the table, and is to be regarded as given to *four places* of decimals. It is tabulated at intervals of 0.0100 from 9.0000 to 0.0000 through the first page of the table, then at intervals of 0.0010, and in the last two divisions at intervals of 0.0001. The characteristic of the argument is printed at the head of the column. The figures supposed to follow the printed figures in the values of the argument are *zeros*. Thus, the first value is 9.0000, the next 9.0100, etc. The angle is given, for convenience of interpolation, in degrees and decimals of a degree. When found, it is easily reduced to degrees and minutes, if that is necessary, and should, in general, be taken *only to the nearest minute*. The angle under the heading $\sin^{-1} u$ is that angle of which the corresponding value of the argument, $\log u$, is the log sin; etc. In interpolating in this table, we may use the *printed rate of difference* instead of Δu, working from the *nearest* tabulated value of the argument, and carefully observing whether the tabulated angle ought to be *increased* or *diminished*. *When the printed rate of difference is omitted*, this is because the interval is too great to admit of accurate interpolation. In this case, we must resort to those later divisions of the table in which the argument is tabulated at smaller intervals. *When the last figure of the tabulated angle is printed in small type*, this shows that that figure is uncertain, if the logarithm is given to only four places; that is, that there is a possible variation, on each side of the tabulated angle, as great as *half a unit* in the place of the figure so printed. For example, if $\log u = 9.9000$, we find the last figures of $\sin^{-1} u$ and $\cos^{-1} u$ to be printed in small type. Now, seven-place tables show that $(\log \sin)^{-1}$ $9.8999500 = 52^\circ.581$, while $(\log \sin)^{-1}\; 9.9000500 = 52^\circ.600$. But 9.9000 may represent any logarithm between these; and hence the corresponding angle, in this case, admits a like variation, while $\cos^{-1} u$ may have any value between $37^\circ.419$ and $37^\circ.400$.

Neither of these difficulties presents itself in finding an angle from its log tan or log ctn. If $\log u = 9.9000$, $\tan^{-1} u$ can only vary from $38°.458$ to $38°.464$.

The angle found by interpolation should be carried out *only to the nearest hundredth of a degree, in any case.* The last column of the table shows that the angle is not always determined even to the nearest *tenth.*

b. If the characteristic of the given logarithm is 0, we must take its *arithmetical complement,* which will be the logarithm of the *reciprocal* function of the same angle. The angle can then be found by the table.

c. If the given logarithm is less than 9.0000, or greater than 0.0000, the tables in the upper part of p. 10 may be used, as explained in § 7, *e*; or pp. 8, 9.

d. Let us find by this table the angles sought above, in § 7, *f.* We have, in the case of the first example,

$$(\log \sec)^{-1}\, 0.0643 = (\log \cos)^{-1}\, 9.9357.$$

Then the table gives

$$\begin{aligned} (\log \cos)^{-1}\, 9.9360 &= 30°.35, \qquad \Delta u = 23,\\ 0.3 \times 0.23 &= .07\\ \therefore\ (\log \cos)^{-1}\, 9.9357 &= 30°.42 = 30°\, 25'. \end{aligned}$$

In fact, the limits of the angle are 30° 24′.2 and 30° 25′.6, the mean value being 30° 24′.9. In this case, the present table gives a better value than the other; but both values are admissible.

In like manner, we have

$$\begin{aligned} (\log \sin)^{-1}\, 9.5663 &= 21°.81 - 20 = 21°.61 = 21°\, 37',\\ (\log \cos)^{-1}\, 9.9188 &= 33°.92 + .04 = 33°.96 = 33°\, 58',\\ (\log \tan)^{-1}\, 0.7507 &= 79°.92 + .02 = 79°.94 = 79°\, 56',\\ (\log \operatorname{ctn})^{-1}\, 0.0496 &= 41°.71 + .03 = 41°.74 = 41°\, 44',\\ (\log \sec)^{-1}\, 0.2272 &= 53°.64 + .02 = 53°.66 = 53°\, 40',\\ (\log \csc)^{-1}\, 0.1433 &= 46°.01 - .04 = 45°.97 = 45°\, 58'. \end{aligned}$$

§ 9. HYPERBOLIC FUNCTIONS.

a. The **hyperbolic functions** are certain functions which bear relations to the equilateral hyperbola similar to those borne by the circular functions to the circle; and they may often be usefully employed both in computation and in analysis. They are named the hyperbolic sine, cosine, tangent, cotangent, secant, and cosecant; and are variously denoted by different writers. They are here represented by the symbols: Sh, Ch, Th, Cth, Sch, Csch. They may be defined by the following formulas, in which

$$\begin{aligned} \epsilon &= \text{the exponential base}\\ &= 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \cdots\\ &= 2.7182818285 \ldots :- \end{aligned}$$

$$\operatorname{Sh} x = \tfrac{1}{2}(\epsilon^{x} - \epsilon^{-x}), \qquad \operatorname{Ch} x = \tfrac{1}{2}(\epsilon^{x} + \epsilon^{-x}), \qquad \operatorname{Th} x = \frac{\operatorname{Sh} x}{\operatorname{Ch} x},$$

$$\operatorname{Cth} x = \frac{1}{\operatorname{Th} x}, \qquad \operatorname{Sch} x = \frac{1}{\operatorname{Ch} x}, \qquad \operatorname{Csch} x = \frac{1}{\operatorname{Sh} x}.$$

They bear to the circular functions the relations expressed by the following formulas, in which $i = \sqrt{-1}$: —

$$\begin{aligned}
\text{Sh}\, x &= \frac{\sin xi}{i}, & \sin x &= \frac{\text{Sh}\, xi}{i},\\
\text{Ch}\, x &= \cos xi, & \cos x &= \text{Ch}\, xi,\\
\text{Th}\, x &= \frac{\tan xi}{i}, & \tan x &= \frac{\text{Th}\, xi}{i},\\
\text{Cth}\, x &= i \,\text{ctn}\, xi, & \text{ctn}\, x &= i\, \text{Cth}\, xi,\\
\text{Sch}\, x &= \sec xi, & \sec x &= \text{Sch}\, xi,\\
\text{Csch}\, x &= i \csc xi, & \csc x &= i\, \text{Csch}\, xi.
\end{aligned}$$

Again, if ϕ is so taken that

$$x = \text{nat log tan}\,(45^\circ + \tfrac{1}{2}\phi),$$

$$\begin{aligned}
\text{Sh}\, x &= \tan \phi, & \text{Ch}\, x &= \sec \phi, & \text{Csch}\, x &= \text{ctn}\, \phi,\\
\text{Th}\, x &= \sin \phi, & \text{Sch}\, x &= \cos \phi, & \text{Cth}\, x &= \csc \phi.
\end{aligned}$$

The value of ϕ determined by this formula has been called by some writers the **Gudermannian** of x, and denoted by the symbol: gd x.

b. From $x = 0.00$ to $x = 1.00$, the function tabulated is gd x in degrees, at intervals of 0.01 in the value of x. The hyperbolic functions of x are then readily found, by the aid of the formulas last given, from the tables of circular functions. Beginning with $x = 1.00$, log Sh x, log Ch x, and log Th x are tabulated, at intervals of 0.01 in the value of x, up to $x = 3.00$, the characteristic of each logarithm being placed at the head of its column; then at intervals of 0.1 up to $x = 6.0$; and lastly at intervals of 1 up to $x = 10.0$. The printed differences are to be used, as in other tables, each through *half the interval* before and after the line on which it stands.

c. If $x > 10$, log Th$x = 0.0000$, while log Shx and log Chx may be found by the formula and table given at the lower right-hand corner of p. 21. The quantity μ is the *modulus* of the denary system of logarithms; that is, it is the denary logarithm of the exponential base. The values of $n\mu$ being given for all integral values of n from 1 to 10, any product $x\mu$ is readily found, by adding together the products of μ by the successive figures of x. Only four decimal-figures should be retained in the result.

d. The functions log Cth x, log Sch x, and log Csch x are the *arithmetical complements* of log Th x, log Ch x, and log Sh x, respectively.

e. The table may be used both directly and inversely. Examples: —

$$\begin{aligned}
&\log \text{Sh}\ 0.5378 &&= \log \tan 29^\circ.427 = 9.7513,\\
&\log \text{Ch}\ 1.5280 &&= 0.3825,\\
&\log \text{Cth}\ 1.8240 &&= 0.0226,\\
&\log \text{Sh}\ 12.5913 &&= \log \text{Ch}\ 12.5913 = 5.1673,\\
&(\log \text{Ch})^{-1}\ 1.6000 &&= 4.377,\\
&(\log \text{Sh})^{-1}\ 5.0000 &&= 12.206,\\
&(\log \text{Th})^{-1}\ 9.9012 &&= 1.089,\\
&(\log \text{Sch})^{-1}\ 9.5873 &&= 1.604,\\
&(\log \text{Csch})^{-1}\ 0.3924 &&= \text{gd}^{-1}\ (\log \tan)^{-1}\ 9.6076\\
&\qquad = \text{gd}^{-1}\ 22^\circ.06 = 0.395.
\end{aligned}$$

§ 10. NATURAL LOGARITHMS.

a. The **natural** system of logarithms is that which is founded on the **exponential base** (see § 9). This number is defined as *the limiting value* to which the expression

$$(1+\epsilon)^{\frac{1}{\epsilon}} = \sqrt[\epsilon]{(1+\epsilon)}$$

approaches, as ϵ approaches 0. It is most frequently denoted by the letter e; but, as being one of the few peculiar constants of analysis, it is here represented by the symbol $\mathfrak{S}$, which may be read "**base.**"

The following formulas are proved in treatises on the Differential Calculus: —

$$\mathfrak{S} = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots,$$

$$\mathfrak{S}^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots,$$

$$\text{nat log }(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots;$$

the second formula being applicable to *all values* of x, but the last only when x is *numerically less than* 1. If x is *very small,* then *approximately*

$$\mathfrak{S}^x = 1 + x, \qquad \text{nat log }(1+x) = x, \qquad \text{nat log }(1-x) = -x.$$

We also have, in the natural system,

$$\log(a+h) = \log a + \log\left(1+\frac{h}{a}\right) = \log a + \frac{h}{a} - \frac{h^2}{2a^2} + \frac{h^3}{3a^3} - \ldots,$$

provided h is numerically less than a.

The *rate of difference* of $\mathfrak{S}^x$, for $\Delta x = 1$, is always $\mathfrak{S}^x$, and that of nat log x is $\frac{1}{x}$.

b. The **numerical value** of $\mathfrak{S}$ or of any power of $\mathfrak{S}$ can be computed, to *any assigned number of decimal-places,* by using a sufficient number of terms of the first two series given above. Thus, to find $\mathfrak{S}$ to four decimal-places, we proceed as follows, observing that, if any term be divided by its number in the series, the next following term is obtained: —

1) 1.00000
2) 1.00000
3) 0.50000
4) 0.16667
5) 0.04167
6) 0.00833
7) 0.00139
8) 0.00020
0.00002

$\mathfrak{S} = 2.7183\ldots\ldots$

c. The **modulus** of any system of logarithms is the logarithm of $\Game$ in that system. If m is the modulus of a system of which a is the base, then

$$a^m = \Game, \qquad \Game^{m^{-1}} = a.$$

The modulus of the *natural system itself* is 1. The values of the modulus of the *denary system* and of the reciprocal of that modulus are

$$\begin{aligned} \mu \;\; &= \text{den log } \Game = 0.4342944819\ldots, \\ \mu^{-1} &= \text{nat log } 10 = 2.3025850930\ldots. \end{aligned}$$

By the rule for converting logarithms from one system to another, the logarithm of a number in any system may be found by *multiplying the modulus of that system into the natural logarithm of the same number.* Thus,

$$\begin{aligned} \text{den log } x &= \mu \text{ nat log } x, \\ \text{nat log } x &= \mu^{-1} \text{ den log } x. \end{aligned}$$

By the aid of these formulas, the table at the bottom of p. 21 may be used to find the natural logarithm of any number, or the denary logarithm of any power of the exponential base, or to find a number from its natural logarithm. For example: —

$$\begin{aligned} \text{nat log } 72.5 \;\;\; &= 1.8603 \times \mu^{-1} = 4.2835, \\ \text{nat log } 1.0074 &= 0.0032 \times \mu^{-1} = 0.0074, \\ \text{den log } \Game^{\frac{1}{7}} \;\;\; &= \tfrac{1}{7}\mu \qquad\qquad\;\; = 0.0620, \end{aligned}$$

$$\begin{aligned} (\text{nat log})^{-1}\, 10.2108 &= (\text{den log})^{-1}\, (10.2108 \times \mu) \\ &= (\text{den log})^{-1}\, 4.4345 = 27194. \end{aligned}$$

d. The natural system is so called, because, in the higher mathematics, it is convenient to regard all other systems as founded upon this. It is named by some writers **hyperbolic**, and by others **Neperian.** But in fact, it is not the system of Napier; nor has it any other relation to the hyperbola than that which belongs to logarithms in general.

e. We may make the following statement of the relation of logarithms and of the hyperbolic functions to the hyperbola, using the notation of Analytic Geometry: —

Let $xy = 1$ be the equation of an hyperbola referred to its asymptotes. It can be proved by the Integral Calculus that the *area*, contained between the curve and the axis of x, and between two ordinates of which one is drawn to the vertex of the curve, is measured by log x in the system of which the modulus is sin ω. Thus, *the logarithms belonging to any system may be represented by the areas of an appropriate hyperbola.* The natural system corresponds to the equilateral hyperbola, for which $\sin \omega = 1$.

Again, if u denotes twice the area of the *sector* of the hyperbola $x^2 - y^2 = 1$, contained between the axis of x and a radius vector from the centre, then

$$x = \text{Ch } u, \qquad y = \text{Sh } u;$$

just as, in the circle $x^2 + y^2 = 1$, with a similar meaning of u,

$$x = \cos u, \qquad y = \sin u.$$

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By H. N. WHEELER, A.M., of Harvard University. 12mo. Cloth. 211 pages. Mailing Price, $1.10; Introduction, $1.00; Allowance, 30 cts.

Peirce's Mathematical Tables, First Series, are bound with this Trigonometry. These Tables are so arranged, and so fully explained, as to be readily intelligible to the beginner in the use of logarithms. Two pages, containing the functions of angles less than 6° and greater than 84°, given *to every minute*, have been added since the publication of the first edition.

Plane Trigonometry. This book originated in a desire to give pupils a better idea of the trigonometric functions of obtuse angles than they could obtain from any book heretofore existing. The author believes that a student can get a comprehensive and thorough knowledge of Trigonometry most quickly and easily, if, at the outset, such definitions are given to the trigonometric functions as will apply to all angles; with this idea for a basis, he has endeavored to prepare an elementary text-book for general use. By beginning with an explanation of the use of the negative sign as applied to lines and angles, and then giving general definitions to the trigonometric functions, he has been able to demonstrate all the fundamental formulas in a perfectly general yet simple manner.

While he has tried to present the subject from an elementary point of view, he has not lost sight of the fact, that, to most students, Trigonometry is merely a stepping-stone to something higher; and for this reason he has also tried to present the results in such a light as will make them effective tools for the student in his future work.

Spherical Trigonometry. The author has endeavored to prepare a book for the use of schools and colleges which, while brief and simple, shall yet be thorough, and suggestive both of the theoretical and practical bearings of the subject.

Such applications to Geometry and Astronomy, and such problems involving these applications, have been given, as will interest the student, and show him that Spherical Trigonometry is not a mere mass of meaningless formulas, but an easy means of solving many practical problems of great importance.

Examples of Differential Equations.

By GEORGE A. OSBORNE, Professor of Mathematics in the Massachusetts Institute of Technology, Boston. 12mo. Cloth. viii + 50 pp. Mailing price, 60 cts.; for Introduction, 50 cts.

Notwithstanding the importance of the study of Differential Equations, either as a branch of pure mathematics, or as applied to Geometry or Physics, no American work on this subject has been published containing a classified series of examples. This book is intended to supply this want, and provides a series of nearly three hundred examples with answers systematically arranged and grouped under the different cases, and accompanied by concise rules for the solution of each case.

It is hoped that the work will be found useful, not only in the study of this important subject, but also by way of reference to mathematical students generally whenever the solution of a differential equation is required.

Elements of the Theory of the Newtonian Potential Function.

By B. O. PEIRCE, Assistant Professor of Mathematics and Physics, Harvard University. 12mo. Cloth. 154 pages. Mailing price, $1.60; for Introduction, $1.50.

A knowledge of the properties of this function is essential for electrical engineers, for students of mathematical physics, and for all who desire more than an elementary knowledge of experimental physics.

This book, based upon notes made for class-room use, was written because no book in English gave in simple form, for the use of students who know something of the calculus, so much of the theory of the potential function as is needed in the study of physics. Both matter and arrangement have been practically adapted to the end in view.

CHAPTER I. The Attraction of Gravitation.
II. The Newtonian Potential Function in the Case of Gravitation.
III. The Newtonian Potential Function in the Case of Repulsive Forces.
IV. Surface Distributions. Green's Theorem.
V. Application of the Results of the Preceding Chapters to Electrostatics.

Determinants.

The Theory of Determinants: an Elementary Treatise. By Paul H. Hanus, B.S., Professor of Mathematics in the University of Colorado. 8vo. Cloth. 000 pages. Mailing price, $0.00; for Introduction, $0.00.

This is a text-book for the use of students in colleges and technical schools. The need of an American work on determinants has long been felt by all teachers and students who have extended their reading beyond the elements of mathematics. The importance of the subject is no longer overlooked. The shortness and elegance imparted to many otherwise tedious processes, by the introduction of determinants, recommend their use even in the more elementary branches, while the advanced student cannot dispense with a knowledge of these valuable instruments of research. Moreover, determinants are employed by all modern writers.

This book is written especially for those who have had no previous knowledge of the subject, and is therefore adapted to self-instruction as well as to the needs of the class-room. To this end the subject is at first presented in a very simple manner. As the reader advances, less and less attention is given to details. Throughout the entire work it is the constant aim to arouse and enliven the reader's interest by first showing how the various concepts have arisen naturally, and by giving such applications as can be presented without exceeding the limits of the treatise. The work is sufficiently comprehensive to enable the student that has mastered the volume to use the determinant notation with ease, and to pursue his further reading in the modern higher algebra with pleasure and profit.

In Chapter I. the evolution of a theory of determinants is touched upon, and it is shown how determinants are produced in the process of eliminating the variables from systems of simple equations with some further preliminary notions and definitions.

In Chapter II. the most important properties of determinants are discussed. Numerous examples serve to fix and exemplify the principles deduced.

Chapter III. comprises half the entire volume. It is the design of this chapter to familiarize the reader with the most important special forms that occur in application, and to enable him to realize the practical usefulness of determinants as instruments of research.

[*Ready June* 1.

Elementary Co-ordinate Geometry.

By W. B. SMITH, Professor of Physics, Missouri State University. 12mo. Cloth. 312 pp. Mailing price, $2.15; for Introduction, $2.00.

While in the study of Analytic Geometry either gain of knowledge or culture of mind may be sought, the latter object alone can justify placing it in a college curriculum. Yet the subject may be so pursued as to be of no great educational value. Mere calculation, or the solution of problems by algebraic processes, is a very inferior discipline of reason. Even geometry is not the best discipline. In all thinking the real difficulty lies in forming clear notions of things. In doing this all the higher faculties are brought into play. It is this formation of concepts, therefore, that is the essential part of mental training. He who forms them clearly and accurately may be safely trusted to put them together correctly. Nearly every seeming mistake in reasoning is really a mistake in conception.

Such considerations have guided the composition of this book. Concepts have been introduced in abundance, and the proofs made to hinge directly upon them. Treated in this way the subject seems adapted, as hardly any other, to develop the power of thought.

Some of the special features of the work are:—

1. Its SIZE is such it can be mastered in the time generally allowed.
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7. The student's grasp is strengthened by numerous EXERCISES.
8. The work has been TESTED at every point IN THE CLASS-ROOM.